Centrally extended mapping class groups from quantum Teichmüller theory*

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Abstract

The central extension of the mapping class groups of punctured surfaces of finite type that arises in quantum Teichmüller theory is 12 times the Meyer class plus the Euler classes of the punctures. This is analogous to the result obtained in [9] for the Thompson groups.

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Introduction

The quantum theory of Teichmüller spaces of punctured surfaces of finite type, originally constructed in [4, 15] and subsequently generalized to higher rank Lie groups and cluster algebras in [7, 8], leads to one parameter families of projective unitary representations of Ptolemy modular groupoids associated to ideal triangulations of punctured surfaces. We will call such representations (quantum) dilogarithmic representations, since the main ingredient in the theory is the non-compact quantum dilogarithm function first introduced in the context of quantum integrable systems by L.D. Faddeev in [5].

These representations are infinite dimensional so that a priori it is not clear if they come from suitable $2+1$-dimensional topological quantum field theories (TQFT). Nonetheless, it is expected that in the singular limit, when the deformation parameter tends to a root of unity[1], the "renormalized" theory corresponds to a finite dimensional TQFT first defined in [13, 14] by using the cyclic representations of the Borel Hopf sub-algebra $BU_q(sl(2))$, and subsequently developed and generalized in [1]. One can get the same finite dimensional representations of Ptolemy modular groupoids directly from compact representations of quantum Teichmüller theory at roots of unity [3, 2, 15].

Projective representations of a group are well known to be equivalent to representations of central extensions of the same group by means of the following procedure. To a group G, a $\mathbb{C}$-vector space V and a group homomorphism $h : G \to PGL(V) \simeq GL(V)/\mathbb{C}^*$, where $\mathbb{C}^*$ is identified with a (normal) subgroup of $GL(V)$ through the imbedding $z \mapsto z\,\mathrm{id}_V$, one can associate a central extension $\widetilde{G}$ of G by a sub-group A of $\mathbb{C}^*$ together with a representation $\widetilde{h} : \widetilde{G} \to GL(V)$ such that the following diagram is commutative and has exact rows:

$$\begin{tikzcd}
1 \arrow[r] & \mathbb{C}^* \arrow[r] & GL(V) \arrow[r] & PGL(V) \arrow[r] & 1 \\
1 \arrow[r] & A \arrow[r] \arrow[u, hook] & \widetilde{G} \arrow[r] \arrow[u, "\widetilde{h}"] & G \arrow[r] \arrow[u, "h"] & 1
\end{tikzcd}$$

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[1] One should distinguish between two different limits, depending on whether $\frac{\log(q)}{2\pi i}$ tends to a positive or a negative rational number. In the case when this limit is a positive rational number, the limit of the representation is non-singular and so it stays infinite dimensional.

One such extension is the pull-back $\widetilde{\mathbf{G}}$ of the central extension $GL(V) \to PGL(V)$ under the homomorphism $G \to PGL(V)$, which is canonically defined. However it is possible to find also smaller extensions associated to proper sub-groups $A \subset \mathbb{C}^*$. The central extension $\widetilde{G}$ associated to the smallest possible sub-group $A \subset \mathbb{C}^*$ for which there exists a linear representation as in the diagram above resolving the projective representation of G will be called the *minimal reduction* of $\widetilde{\mathbf{G}}$.

In this light, quantum Teichmüller theory gives rise to representations of certain central extensions of the surface mapping class groups which are the vertex groups of the Ptolemy modular groupoids. The main goal of this paper is to identify the isomorphism classes of those central extensions. Namely, by using the quantization approach of [15], we extend the analysis of the particular case of a once punctured genus three surface performed in [16] to arbitrary punctured surfaces of finite type.

Let a group G with a given presentation be identified as the quotient group F/R, where F is a free group and R, the normal subgroup generated by the relations. Then, a central extension of G can be obtained from a homomorphism $\overline{h}\colon F \to GL(V)$ with the property $\overline{h}(R) \subset \mathbb{C}^*$ so that it induces a homomorphism $h\colon G \to PGL(V)$. In this case, the homomorphism $\overline{h}$ will be called an *almost linear representation* of G, in order to distinguish it from a projective representation.

In quantum Teichmüller theory, central extensions of surface mapping class groups appear through almost linear representations. Specifically, let $\Gamma^s_{g,r}$ be the mapping class group of a surface $\Sigma^s_{g,r}$ of genus g with r boundary components and s punctures. These are mapping classes of homeomorphisms which fix the boundary point-wise and fix the set of punctures (not necessarily point-wise). Denoting $\Gamma^s_g = \Gamma^s_{g,0}$, the projective representations of Γ^s_g for $(2g-2+2s)s > 0$, constructed in [15, 16], are almost linear representations corresponding to certain central extensions $\widetilde{\Gamma}^s_g$. By considering embeddings $\Sigma^s_{g,r} \subset \Sigma^t_{h,0}$, the central extensions $\widetilde{\Gamma}^s_g$ can be used to define central extensions for the mapping class groups $\Gamma^s_{g,r}$ for $s \geq r$, and the associated surfaces containing on each boundary component at least one puncture. According to [21], any embedding $\Sigma^s_{g,r} \subset \Sigma^t_{h,0}$, for which $\Sigma^t_h \setminus \Sigma^s_{g,r}$ contains no disk or cylinder components, induces an embedding of the corresponding mapping class groups. Using this fact, we can define the central extension $\widetilde{\Gamma}^s_{g,r}$ as the pull-back of the central extension $\widetilde{\Gamma}^t_h$ by the injective homomorphism $\Gamma^s_{g,r} \hookrightarrow \Gamma^t_h$ induced by an embedding of the corresponding surfaces. A priori, it is not clear whether such definition depends on a particular choice of the embedding, but our main result below shows that this is indeed the case.

Central extensions by an Abelian group A of a given group G are known to be classified, up to isomorphism, by elements of the 2-cohomology group $H^2(G; A)$. In the case of surface mapping class groups $\Gamma^s_{g,r}$, the latter was first computed by Harer in [12] for $g \geq 5$ and further completed by Korkmaz and Stipsicz in [18] for $g \geq 4$ (see also [17] for a survey). Specifically, we have

$$H^2(\Gamma^s_{g,r}) = \mathbb{Z}^{s+1}, \text{ if } g \geq 4,$$

where the generators are given by (one fourth of) the Meyer signature class χ (it is the only generator for the groups $H^2(\Gamma_g) \cong H^2(\Gamma_{g,1}) \simeq \mathbb{Z}$, see [20, 12, 18] for its definition) and s Euler classes e_i associated with s punctures. In the case when $g = 3$, the group $H^2(\Gamma^s_{3,r})$ still contains the sub-group $\mathbb{Z}^{s+1}$ generated by the above mentioned classes, but it is not known whether there are other (2-torsion) classes. When $g = 2$ we will show that $H^2(\Gamma^s_{2,r})$ contains the subgroup $\mathbb{Z}/10\mathbb{Z} \oplus \mathbb{Z}^s$, whose torsion part is generated by χ and whose free part is generated by the Euler classes. The Universal Coefficients Theorem permits then to compute $H^2(G; A)$ for every Abelian group A.

Denote as above by $\widetilde{\mathbf{\Gamma^s_{g,r}}}$ the canonical central extension of $\Gamma^s_{g,r}$ by $\mathbb{C}^*$ which is obtained as the pull-back of the canonical central extension $GL(\mathcal{H}) \to PGL(\mathcal{H})$ under the quantum projective representation associated to a semi-symmetric T in the Hilbert space $\mathcal{H}$ (see the next section). Quantum representations depend on some parameter $\zeta \in \mathbb{C}^*$. Our main result is the following theorem.

Theorem 0.1. *The central extension $\widetilde{\mathbf{\Gamma^s_{g,r}}}$ can be reduced to a minimal central extension $\widetilde{\Gamma}^s_{g,r}$ of $\Gamma^s_{g,r}$ by a cyclic Abelian $A \subset C^*$, where A is the subgroup of C^* generated by ζ^{-6}. Moreover its cohomology class is*

$$c_{\widetilde{\Gamma}^s_{g,r}} = 12\chi + \sum_{i=1}^{s} e_i \in H^2(\Gamma^s_{g,r}; A)$$

if $g \geq 2$ and $s \geq 4$. Here χ and e_i are one fourth of the Meyer signature class and respectively the i-th Euler class with A coefficients.

There is a geometric interpretation of this extension.

Corollary 0.2. *Let us consider the extension $\widehat{\Gamma_{g,r+s}}$ of class 12χ. Then there is an exact sequence:*

$$1 \to A^{s-1} \to \widehat{\Gamma_{g,r+s}} \to \widetilde{\Gamma^s_{g,r}} \to 1$$

In some sense the quantum representations of punctured mapping class groups can be lifted to the mapping class groups of surfaces with boundary obtained by blowing up the punctures.

Corollary 0.3. *The cohomology class of the central extension $\widetilde{\mathbf{\Gamma^s_{g,r}}}$ is*

$$c_{\widetilde{\mathbf{\Gamma^s_{g,r}}}} = 12\chi + \sum_{i=1}^{s} e_i \in H^2(\Gamma^s_{g,r}; \mathbb{C}^*)$$

if $g \geq 3$ and $s \geq 4$. The same formula holds also when $g = 2$ but the class χ vanishes in $H^2(\Gamma^s_{g,r}; \mathbb{C}^*)$. Here χ and e_i are one fourth of the Meyer signature class and respectively the i-th Euler class with $\mathbb{C}^*$ coefficients.

Remark 0.1. The central extension arising from $SU(2)$-TQFT with p_1-structures was computed in [10, 19] for Γ_g and it equals 12χ. It can be shown that their computations extend to the case of punctured surfaces and the associated class for $\Gamma^s_{g,r}$ is $12\chi + \sum_{i=1}^{s} e_i$. Our result shows that this extension coincides with the central extension arising from quantum Teichmüller theory.

The organization of the paper is as follows. In Section 1, we review the quantization of the Teichmüller space of a punctured surface and define the associated quantum representations of the decorated Ptolemy groupoid which correspond to linear representations of a central extension of the decorated Ptolemy groupoid. Then, in Section 2, we prove Theorem 0.1 by finding the pull-back of this central extension to the mapping class group of the surface. The key idea is to use a Grothendieck type principle. Namely, one can identify a central extension of the mapping class group of some surface, if one understands its restrictions to the mapping class groups of sub-surfaces of bounded topological types. The core of the proof consists in computing explicitly the lifts to the central extension of the decorated Ptolemy groupoid of the relations known to hold in the mapping class groups. When properly interpreted, these lifts yield the class of the mapping class group extension.

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1 Quantum Teichmüller theory

1.1 The groupoid of decorated ideal triangulations

Let $\Sigma = \Sigma^s_g$ be an oriented closed surface of genus g with s punctures. Denote $M = 2g - 2 + s$ and assume that $Ms > 0$. Then surface Σ admits ideal triangulations with vertices at the s punctures.

Definition 1.1. *A* decorated ideal triangulation *of Σ is an ideal triangulation τ, where all triangles are provided with a marked corner, and a bijective ordering map*

$$\bar{\tau}\colon \{1, \ldots, 2M\} \ni j \mapsto \bar{\tau}_j \in T(\tau)$$

is fixed. Here $T(\tau)$ is the set of all triangles of τ.

Graphically, the marked corner of a triangle is indicated by an asterisk and the corresponding number is put inside the triangle. The set of all decorated ideal triangulations of Σ is denoted Δ_Σ.

Recall that if a group G freely acts in a set X, then there is an associated groupoid defined as follows. The objects are the G-orbits in X, while morphisms are G-orbits in $X \times X$ with respect to the diagonal action. Denote by $[x]$ the object represented by an element $x \in X$ and $[x, y]$ the morphism represented by a pair of elements $(x, y) \in X \times X$. Two morphisms $[x, y]$ and $[u, v]$, are composable if and only if $[y] = [u]$ and their composition is $[x, y][u, v] = [x, gv]$, where $g \in G$ is the unique element sending u to y. The inverse and the identity morphisms are given respectively by $[x, y]^{-1} = [y, x]$ and $\mathrm{id}_{[x]} = [x, x]$. In what follows, products of the form $[x_1, x_2][x_2, x_3] \cdots [x_{n-1}, x_n]$ will be shortened as $[x_1, x_2, x_3, \ldots, x_{n-1}, x_n]$.

The mapping class group $\mathcal{M}_\Sigma$ of Σ acts freely on Δ_Σ. In this case, we let $\mathcal{G}_\Sigma$ denote the corresponding groupoid, called the *groupoid of decorated ideal triangulations*, or *decorated Ptolemy groupoid*. There is a presentation for $\mathcal{G}_\Sigma$ with three types of generators and four types of relations.

The generators are of the form $[\tau, \tau^\sigma]$, $[\tau, \rho_i\tau]$, and $[\tau, \omega_{i,j}\tau]$, where τ^σ is obtained from τ by replacing the ordering map $\bar\tau$ by the map $\bar\tau \circ \sigma$, where $\sigma \in S_{2M}$ is a permutation of the set $\{1, \dots, 2M\}$, $\rho_i\tau$ is obtained from τ by changing the marked corner of triangle $\bar\tau_i$ as in Figure 1, and $\omega_{i,j}\tau$ is obtained from τ by applying the flip transformation in the quadrilateral composed of triangles $\bar\tau_i$ and $\bar\tau_j$ as in Figure 2.

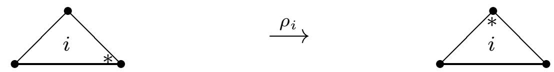


Figure 1: The transformation ρ_i.

Figure 2: The transformation $\omega_{i,j}$.

There are two sets of relations satisfied by these generators. The first set is as follows:

$$[\tau, \tau^\alpha, (\tau^\alpha)^\beta] = [\tau, \tau^{\alpha\beta}], \quad \alpha, \beta \in S_{2M}, \tag{1}$$

$$[\tau, \rho_i\tau, \rho_i\rho_i\tau, \rho_i\rho_i\rho_i\tau] = \mathrm{id}_{[\tau]}, \tag{2}$$

$$[\tau, \omega_{ij}\tau, \omega_{ik}\omega_{ij}\tau, \omega_{jk}\omega_{ik}\omega_{ij}\tau] = [\tau, \omega_{jk}\tau, \omega_{ij}\omega_{jk}\tau], \tag{3}$$

$$[\tau, \omega_{ij}\tau, \rho_i\omega_{ij}\tau, \omega_{ji}\rho_i\omega_{ij}\tau] = [\tau, \tau^{(ij)}, \rho_j\tau^{(ij)}, \rho_i\rho_j\tau^{(ij)}], \tag{4}$$

where the first two relations are evident, while the other two are shown graphically in Figures 3, 4.

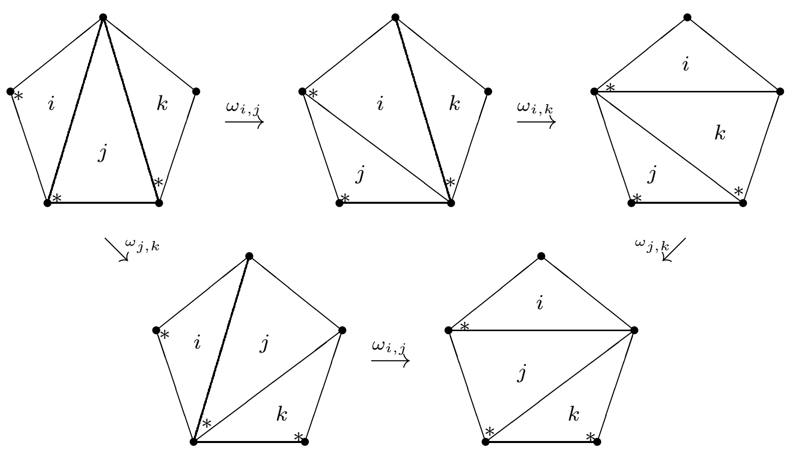


Figure 3: The Pentagon relation (3).

The following commutation relations fulfill the second set of relations:

$$[\tau, \rho_i\tau, (\rho_i\tau)^\sigma] = [\tau, \tau^\sigma, \rho_{\sigma^{-1}(i)}\tau^\sigma], \tag{5}$$

$$[\tau, \omega_{ij}\tau, (\omega_{ij}\tau)^\sigma] = [\tau, \tau^\sigma, \omega_{\sigma^{-1}(i)\sigma^{-1}(i)}\tau^\sigma], \tag{6}$$

$$[\tau, \rho_j\tau, \rho_i\rho_j\tau] = [\tau, \rho_i\tau, \rho_j\rho_i\tau], \tag{7}$$

$$[\tau, \rho_i\tau, \omega_{jk}\rho_i\tau] = [\tau, \omega_{jk}\tau, \rho_i\omega_{jk}\tau], \ \ i \notin \{j, k\}, \tag{8}$$

$$[\tau, \omega_{ij}\tau, \omega_{kl}\omega_{ij}\tau] = [\tau, \omega_{kl}\tau, \omega_{ij}\omega_{kl}\tau], \ \ \{i, j\} \cap \{k, l\} = \emptyset. \tag{9}$$

1.2 Hilbert spaces of square integrable functions associated to triangulations

In what follows, we work with Hilbert spaces

$$\mathcal{H} \equiv L^2(\mathbb{R}), \quad \mathcal{H}^{\otimes n} \equiv L^2(\mathbb{R}^n).$$

Any two self-adjoint operators p and q, acting in $\mathcal{H}$ and satisfying the Heisenberg commutation relation

$$\mathsf{pq} - \mathsf{qp} = (2\pi\mathsf{i})^{-1}\,\mathrm{id}_{\mathcal{H}}, \tag{10}$$

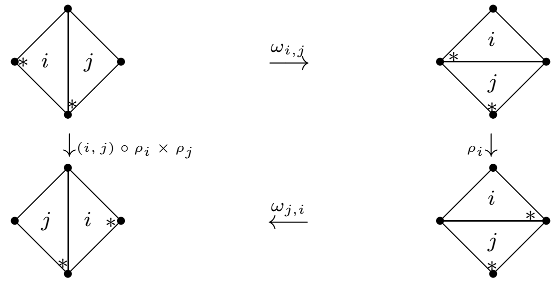


Figure 4: The Inversion relation (4).

can be realized as differentiation and multiplication operators. Such "coordinate" realization in Dirac's bra-ket notation has the form

$$\langle x|\mathsf{p} = \frac{1}{2\pi\mathsf{i}}\frac{\partial}{\partial x}\langle x|, \quad \langle x|\mathsf{q} = x\langle x|. \tag{11}$$

Formally, the set of "vectors" $\{|x\rangle\}_{x\in\mathbb{R}}$ forms a generalized basis of $\mathcal{H}$ with the following orthogonality and completeness properties:

$$\langle x|y\rangle = \delta(x-y), \qquad \int_{\mathbb{R}} |x\rangle dx\langle x| = \mathrm{id}_{\mathcal{H}}\,.$$

For any $1 \le i \le m$ we shall use the following notation

$$\iota_i\colon \ \mathrm{End}\,\mathcal{H} \ni \mathsf{a} \mapsto \mathsf{a}_i = \underbrace{1\otimes\cdots\otimes 1}_{i-1\ \text{times}}\otimes\mathsf{a}\otimes 1\otimes\cdots\otimes 1 \in \mathrm{End}\,\mathcal{H}^{\otimes m}.$$

Besides that, if $\mathsf{u} \in \mathrm{End}\,\mathcal{H}^{\otimes k}$ for some $1 \le k \le m$ and $\{i_1, i_2, \ldots, i_k\} \subset \{1, 2, \ldots, m\}$, then we shall write

$$\mathsf{u}_{i_1 i_2 \ldots i_2} \equiv \iota_{i_1}\otimes\iota_{i_2}\otimes\cdots\otimes\iota_{i_k}(\mathsf{u}).$$

The symmetric group S_m naturally acts in $\mathcal{H}^{\otimes m}$:

$$\mathsf{P}_\sigma(x_1\otimes\cdots\otimes x_i\otimes\cdots\otimes x_m) = x_{\sigma^{-1}(1)}\otimes\cdots\otimes x_{\sigma^{-1}(i)}\otimes\ldots\otimes x_{\sigma^{-1}(m)}, \quad \sigma\in S_m. \tag{12}$$

1.3 Semi-symmetric T-matrices

We define now the algebraic structure needed for constructing representations of the decorated Ptolemy groupoid $\mathcal{G}_\Sigma$.

Definition 1.2. *A* semi-symmetric T-matrix *consists of two operators* $\mathsf{A} \in \mathrm{End}(\mathcal{H})$ *and* $\mathsf{T} \in \mathrm{End}(\mathcal{H}^{\otimes 2})$ *satisfying the equations:*

$$\mathsf{A}^3 = 1, \tag{13}$$
$$\mathsf{T}_{12}\mathsf{T}_{13}\mathsf{T}_{23} = \mathsf{T}_{23}\mathsf{T}_{12}, \tag{14}$$
$$\mathsf{T}_{12}\mathsf{A}_1\mathsf{T}_{21} = \zeta\mathsf{A}_1\mathsf{A}_2\mathsf{P}_{(12)}, \tag{15}$$

where $\zeta\in\mathbb{C}^*$ *and the permutation operator* $\mathsf{P}_{(12)}$ *is defined by equation* (12)*, for* σ *denoting the transposition* (12)*.*

Examples of semi-symmetric T-matrices could be obtained as follows. Fix some self-conjugate operators p, q satisfying the Heisenberg commutation relation(10). Choose a parameter b satisfying the condition:

$$(1-|b|)\mathrm{Im}b = 0,$$

and define then two unitary operators by the following formulas:

$$\mathsf{A} \equiv e^{-\mathsf{i}\pi/3}e^{\mathsf{i}3\pi\mathsf{q}^2}e^{\mathsf{i}\pi(\mathsf{p}+\mathsf{q})^2} \in \mathrm{End}(\mathcal{H}), \tag{16}$$
$$\mathsf{T} \equiv e^{\mathsf{i}2\pi\mathsf{p}_1\mathsf{q}_2}\varphi_b(\mathsf{q}_1+\mathsf{p}_2-\mathsf{q}_2) \in \mathrm{End}(\mathcal{H}^{\otimes 2}). \tag{17}$$

They satisfy the defining relations for a semi-symmetric T-matrix, where

$$\zeta = e^{\mathsf{i}\pi c_b^2/3}, \quad c_b = \frac{\mathsf{i}}{2}(b+b^{-1}), \tag{18}$$

and φ_b is the Faddeev non-compact quantum logarithm defined on $\{z \in \mathbb{C}; |\mathrm{Im(z)}| < |\mathrm{Im(c_b)}|\}$ by means of

$$\varphi_b(z) = \exp\left(-\frac{1}{4}\int_{-\infty}^{\infty} \frac{\exp(-2\pi i z x)d\,x}{\sinh(xb)\sinh(x/b)x}\right) \tag{19}$$

Its main feature is the following functional equation it satisfies:

$$\varphi_b(q)\varphi_b(p) = \varphi_b(p)\varphi_b(p+q)\varphi_b(q)$$

whenever $pq - qp = \frac{1}{2\pi i}\mathbf{1}$.
Remark that the operator A is characterized (up to a normalization factor) by the equations:

$$\mathsf{A}\mathsf{q}\mathsf{A}^{-1} = \mathsf{p} - \mathsf{q}, \quad \mathsf{A}\mathsf{p}\mathsf{A}^{-1} = -\mathsf{q}.$$

Note that equations (13)—(15) correspond to relations (2)—(4).

Let us introduce now some notation which will be useful in the sequel. For any operator $\mathsf{a} \in \mathrm{End}\,\mathcal{H}$ we set:

$$\mathsf{a}_{\hat{k}} \equiv \mathsf{A}_k \mathsf{a}_k \mathsf{A}_k^{-1}, \quad \mathsf{a}_{\check{k}} \equiv \mathsf{A}_k^{-1}\mathsf{a}_k\mathsf{A}_k. \tag{20}$$

It is evident that

$$\mathsf{a}_{\check{\hat{k}}} = \mathsf{a}_{\hat{\check{k}}} = \mathsf{a}_k, \quad \mathsf{a}_{\hat{\hat{k}}} = \mathsf{a}_{\check{k}}, \quad \mathsf{a}_{\check{\check{k}}} = \mathsf{a}_{\hat{k}},$$

where the last two equations follow from equation (13). In particular, we have

$$\mathsf{p}_{\hat{k}} = -\mathsf{q}_k, \quad \mathsf{q}_{\hat{k}} = \mathsf{p}_k - \mathsf{q}_k, \tag{21}$$

$$\mathsf{p}_{\check{k}} = \mathsf{q}_k - \mathsf{p}_k, \quad \mathsf{q}_{\check{k}} = -\mathsf{p}_k. \tag{22}$$

Besides that, it will be also useful to use the notation

$$\mathsf{P}_{(kl\ldots m\hat{k})} \equiv \mathsf{A}_k\mathsf{P}_{(kl\ldots m)}, \quad \mathsf{P}_{(kl\ldots m\check{k})} \equiv \mathsf{A}_k^{-1}\mathsf{P}_{(kl\ldots m)}, \tag{23}$$

where $(kl\ldots m)$ is the cyclic permutation

$$(kl\ldots m)\colon k \mapsto l \mapsto \ldots \mapsto m \mapsto k.$$

Equation (15) in this notation takes a rather compact form

$$\mathsf{T}_{12}\mathsf{T}_{2\hat{1}} = \zeta\mathsf{P}_{(12\hat{1})}. \tag{24}$$

Remark 1.1. One can derive the following symmetry property of the T-matrix: $\mathsf{T}_{12} = \mathsf{T}_{\hat{2}\check{1}}$.

1.4 The quantum Teichmüller space

The quantization of the Teichmüller space of a punctured surface Σ induced by a semi-symmetric T-matrix is defined by means of a *quantum functor*:

$$\mathsf{F}\colon \mathcal{G}_\Sigma \to \mathrm{End}(\mathcal{H}^{\otimes 2M}),$$

Its meaning is that we have a operator valued function:

$$\mathsf{F}\colon \Delta_\Sigma \times \Delta_\Sigma \to \mathrm{End}(\mathcal{H}^{\otimes 2M}),$$

satisfying the following equations:

$$\mathsf{F}(\tau,\tau) = \mathrm{id}_{\mathcal{H}^{\otimes 2M}}, \quad \mathsf{F}(\tau,\tau')\mathsf{F}(\tau',\tau'')\mathsf{F}(\tau'',\tau) \in \mathbb{C}\setminus\{0\}, \quad \forall \tau,\tau',\tau'' \in \Delta_\Sigma, \tag{25}$$

$$\mathsf{F}(f(\tau), f(\tau')) = \mathsf{F}(\tau,\tau'), \qquad \forall f \in \mathcal{M}_\Sigma, \tag{26}$$

$$\mathsf{F}(\tau, \rho_i\tau) \equiv \mathsf{A}_i, \tag{27}$$

$$\mathsf{F}(\tau, \omega_{i,j}\tau) \equiv \mathsf{T}_{ij}, \tag{28}$$

$$\mathsf{F}(\tau, \tau^\sigma) \equiv \mathsf{P}_\sigma, \quad \forall \sigma \in S_{2M}, \tag{29}$$

where operator P_σ is defined by equation (12). Consistency of these equations is ensured by the consistency of equations (13)—(15) with relations (2)—(4).

A particular case of equation (25) corresponds to $\tau'' = \tau$:

$$\mathsf{F}(\tau, \tau')\mathsf{F}(\tau', \tau) \in \mathbb{C} \setminus \{0\}. \tag{30}$$

As an example, we can calculate the operator $\mathsf{F}(\tau, \omega_{i,j}^{-1}(\tau))$. Denoting $\tau' \equiv \omega_{i,j}^{-1}(\tau)$ and using equation (30), as well as definition (28), we obtain

$$\mathsf{F}(\tau, \omega_{i,j}^{-1}(\tau)) = \mathsf{F}(\omega_{i,j}(\tau'), \tau') \simeq (\mathsf{F}(\tau', \omega_{i,j}(\tau')))^{-1} = \mathsf{T}_{ij}^{-1}, \tag{31}$$

where $\simeq$ means equality up to a numerical multiplicative factor.

The quantum functor induces a unitary projective representation of the mapping class group $\mathcal{M}_\Sigma$ as follows:

$$\mathcal{M}_\Sigma \ni f \mapsto \mathsf{F}(\tau, f(\tau)) \in \mathrm{End}(\mathcal{H}^{\otimes 2M}).$$

Indeed, we have the following relation (up to a non-zero scalar):

$$\mathsf{F}(\tau, f(\tau))\mathsf{F}(\tau, h(\tau)) = \mathsf{F}(\tau, f(\tau))\mathsf{F}(f(\tau), f(h(\tau))) \simeq \mathsf{F}(\tau, fh(\tau)).$$

The main question addressed in this present paper is to identify the central extension of the mapping class group corresponding to this projective representation. Observe that the projective factor lies in the sub-group of $\mathbb{C}^*$ generated by ζ.

2 Presentation of $\widetilde{\Gamma}_{g,r}^s$

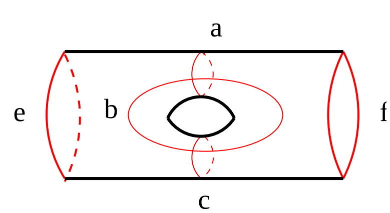
2.1 Generating set for the relations

We start with a number of notations and definitions. Let us denote $\Sigma_{g,r} = \Sigma_{g,r}^0$.

Definition 2.1. *A chain relation C on the surface $\Sigma_{g,r}^s$ is given by an embedding $\Sigma_{1,2} \subset \Sigma_{g,r}^s$ and the standard chain relation on this 2-holed torus, namely*

$$(D_a D_b D_c)^4 = D_e D_f$$

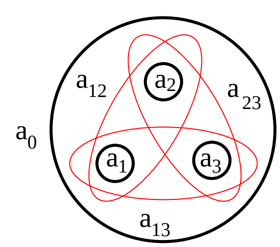
where a, b, c, d, e, f are the following curves of the embedded 2-holed torus:



Definition 2.2. *A lantern relation L on the surface $\Sigma_{g,r}^s$ is given by an embedding $\Sigma_{0,4} \subset \Sigma_{g,r}^s$ and the standard lantern relation on this 4-holed sphere, namely*

$$D_{a_{12}} D_{a_{13}} D_{a_{23}} D_{a_0}^{-1} D_{a_1}^{-1} D_{a_2}^{-1} D_{a_3}^{-1} = 1 \tag{32}$$

where $a_0, a_1, a_2, a_3, a_{12}, a_{13}, a_{23}$ are the following curves of the embedded 4-holed sphere:



Definition 2.3. *Consider an embedding $\Sigma_{0,3}^1 \subset \Sigma_{g,r}^s$ such that the boundary components a_1, a_2, a_3 of $\Sigma_{0,3}^1$ are non-separating curves. Let then a_{12}, a_{13}, a_{23} be embedded curves on $\Sigma_{0,3}^1$ so that a_{jk} bounds a pair of pants $\Sigma_{0,3} \subset \Sigma_{0,3}^1$ along with a_j and a_k, for all $1 \leq j \neq k \leq 3$. Then the puncture relation P (supported at the puncture of $\Sigma_{0,3}^1$) on the surface $\Sigma_{g,r}^s$ is:*

$$D_{a_{12}} D_{a_{13}} D_{a_{23}} D_{a_1}^{-1} D_{a_2}^{-1} D_{a_3}^{-1} = 1 \tag{33}$$

Remark 2.1. The puncture relation is, in fact, a consequence of the lantern relation and the fact that the Dehn twist along a small loop encircling a puncture is trivial.

The first step in proving Theorem 0.1 is to find an explicit presentation for the central extension $\widetilde{\Gamma}_{g,r}^s$. Specifically, by using Gervais' presentation [10], we have the following description.

Proposition 2.1. *Suppose that $g \geq 2$ and $s \geq 4$. Then the group $\widetilde{\Gamma}_{g,r}^s$ has the following presentation.*

1. *Generators:*

   (a) *With each non-separating simple closed curve a in $\Sigma_{g,r}^s$ is associated a generator $\widetilde{D}_a$;*

   (b) *One (central) element z.*

2. *Relations:*

   (a) *Centrality:*
$$z\widetilde{D}_a = \widetilde{D}_a z \tag{34}$$
   for any non-separating simple closed curve a on $\Sigma_{g,r}^s$;

   (b) *Braid type* 0*-relations:*
$$\widetilde{D}_a\widetilde{D}_b = \widetilde{D}_b\widetilde{D}_a \tag{35}$$
   for each pair of disjoint non-separating simple closed curves a and b;

   (c) *Braid type* 1*-relations:*
$$\widetilde{D}_a\widetilde{D}_b\widetilde{D}_a = \widetilde{D}_b\widetilde{D}_a\widetilde{D}_b \tag{36}$$
   for each pair of non-separating simple closed curves a and b which intersect transversely at one point;

   (d) *One lantern relation on a* 4*-holed sphere subsurface with non-separating boundary curves:*
$$\widetilde{D}_{a_0}\widetilde{D}_{a_1}\widetilde{D}_{a_2}\widetilde{D}_{a_3} = \widetilde{D}_{a_{12}}\widetilde{D}_{a_{13}}\widetilde{D}_{a_{23}} \tag{37}$$

   (e) *One chain relation on a* 2*-holed torus subsurface with non-separating boundary curves:*
$$(\widetilde{D}_a\widetilde{D}_b\widetilde{D}_c)^4 = z^{12}\widetilde{D}_e\widetilde{D}_f \tag{38}$$

   (f) *Puncture relations:*
$$\widetilde{D}_{a_{12(i)}}\widetilde{D}_{a_{13}(i)}\widetilde{D}_{a_{23}(i)} = z\widetilde{D}_{a_1(i)}\widetilde{D}_{a_2(i)}\widetilde{D}_{a_3(i)} \tag{39}$$
   for each puncture p_i of $\Sigma_{g,r}^s$, $i \in \{1, 2, \dots, s\}$.

   (g) *Scalar equation:*
$$z^N = 1 \tag{40}$$
   where N is the order of ζ^{-6}, in the case where $\zeta \in \mathbb{C}^*$ is a root of unity.

2.2 Proof of Proposition 2.1

Lemma 2.1. *For any lifts $\widetilde{D}_a$ of the Dehn twists D_a we have $\widetilde{D}_a\widetilde{D}_b = \widetilde{D}_b\widetilde{D}_a$ and thus relations (2) are satisfied.*

Proof. The commutativity relations are satisfied for particular lifts coming from a semi-symmetric T-matrix. If we change the lifts by multiplying each lift by some central element the commutativity is still valid. Thus, the commutativity holds for any lifts. □

Lemma 2.2. *There are lifts $\widetilde{D}_a$ of the Dehn twists D_a, for each non-separating simple closed curve a such that we have $\widetilde{D}_a\widetilde{D}_b\widetilde{D}_a = \widetilde{D}_b\widetilde{D}_a\widetilde{D}_b$ for any simple closed curves a, b with one intersection point, and thus the braid type* 1*-relations (3) are satisfied.*

Proof. Consider an arbitrary lift of one braid type 1-relation (to be called the fundamental one), which has the form $\widetilde{D}_a\widetilde{D}_b\widetilde{D}_a = z^k\widetilde{D}_b\widetilde{D}_a\widetilde{D}_b$. Change then the lift $\widetilde{D}_b$ into $z^k\widetilde{D}_b$. With the new lift the relation above becomes $\widetilde{D}_a\widetilde{D}_b\widetilde{D}_a = \widetilde{D}_b\widetilde{D}_a\widetilde{D}_b$.

Choose now an arbitrary braid type 1-relation of $\Gamma^s_{g,r}$, say $D_xD_yD_x = D_yD_xD_y$. There exists a 1-holed torus $\Sigma_{1,1} \subset \Sigma^s_{g,r}$ containing x, y, namely a neighborhood of $x\cup y$. Let T be the similar torus containing a, b. Since a, b and x, y are non-separating there exists a homeomorphism $\varphi : \Sigma^s_{g,r} \to \Sigma^s_{g,r}$ such that $\varphi(a) = x$ and $\varphi(b) = y$. We have then

$$D_x = \varphi D_a\varphi^{-1}, \quad D_y = \varphi D_b\varphi^{-1}.$$

Let us consider now an arbitrary lift $\widetilde{\varphi}$ of φ, which is well-defined only up to a central element, and set

$$\widetilde{D}_x = \widetilde{\varphi}\widetilde{D}_a\widetilde{\varphi}^{-1}, \quad \widetilde{D}_y = \widetilde{\varphi}\widetilde{D}_b\widetilde{\varphi}^{-1}.$$

These lifts are well-defined since they do not depend on the choice of $\widetilde{\varphi}$ (the central elements coming from $\widetilde{\varphi}$ and $\widetilde{\varphi}^{-1}$ mutually cancel). Moreover, we have then

$$\widetilde{D}_x\widetilde{D}_y\widetilde{D}_x = \widetilde{D}_y\widetilde{D}_x\widetilde{D}_y$$

and so the braid type 1-relations (3) are all satisfied. □

Lemma 2.3. *The choice of lifts of all $\widetilde{D}_x$, with x non-separating, satisfying the requirements of Lemma 2.2 is uniquely defined by fixing the lift $\widetilde{D}_a$ of one particular Dehn twist.*

Proof. In fact the choice of $\widetilde{D}_a$ fixes the choice of $\widetilde{D}_b$. If x is a non-separating simple closed curve on $\Sigma^s_{g,r}$, then there exists another non-separating curve y which intersects it in one point. Thus, by Lemma 2.2, the choice of $\widetilde{D}_x$ is unique. □

Lemma 2.4. *One can choose the lifts of Dehn twists in $\widetilde{\Gamma}^s_{g,r}$ so that all braid type relations are satisfied and the lift of the lantern relation is trivial, namely*

$$\widetilde{D}_a\widetilde{D}_b\widetilde{D}_c\widetilde{D}_d = \widetilde{D}_u\widetilde{D}_v\widetilde{D}_w$$

for the non-separating curves on an embedded $\Sigma_{0,4} \subset \Sigma^s_{g,r}$.

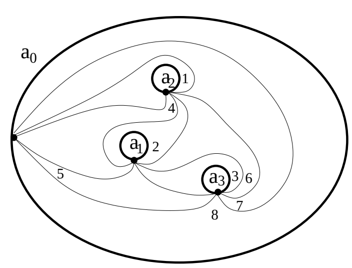
Proof. Arbitrary lift of that lantern relation is of the form $\widetilde{D}_a\widetilde{D}_b\widetilde{D}_c\widetilde{D}_d = z^k\widetilde{D}_u\widetilde{D}_v\widetilde{D}_w$. In this case, we change the lift $\widetilde{D}_a$ into $z^{-k}\widetilde{D}_a$ and adjust the lifts of all other Dehn twists along non-separating curves the way that all braid type 1-relations are satisfied. Then, the required form of the lantern relation is satisfied. □

We say that the lifts of the Dehn twists are *normalized* if all braid type relations and one lantern relation are lifted in a trivial way.

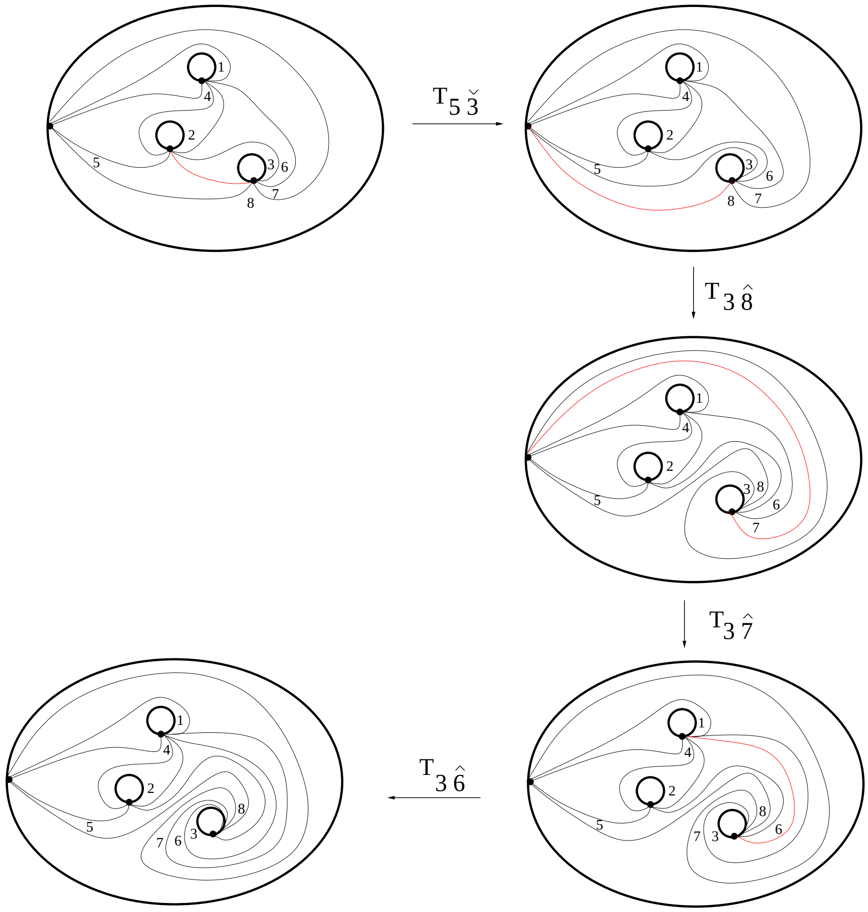
Lemma 2.5. *A normalized Dehn twist in quantum Teichmüller theory is conjugated to the inverse T-matrix times ζ^{-6} i.e.*

$$\widetilde{D}_\alpha = \mathsf{F}(\tau, D_\alpha\tau) = \zeta^{-6}U_\alpha T^{-1}_{kl}U^{-1}_\alpha.$$

Proof. The idea of the proof is to calculate the lift of the lantern relation. Consider the following decorated triangulation τ of the 4-holed disk with 4 punctures:



The trick used in [15, 16] for computing D_a is to use a sequence of flips to change the triangulation into one which intersects some curve isotopic to a into two points. Then the Dehn twist along a can be expressed as the flip of one of the two edges of the latter triangulation intersecting a. This recipe generalizes to the case where the curve a intersects several edges of the triangulation, if a is a boundary component with one puncture on it. Specifically, let $e_1, \ldots, e_s$ be the edges issued from the puncture, in counterclockwise order. Then the Dehn twist D_a can be expressed as the result of composing the flips of $e_1, e_2, \ldots, e_{s-1}$. We illustrate this procedure with the case of the left Dehn twist $D^{-1}_{a_3}$ on the triangulation τ above:



In particular, we find that the following expression for the right Dehn twist along a_3:

$$\bar{\mathsf{F}}_{a_3} = \bar{\mathsf{F}}(\tau, D_{a_3}\tau) = T_{3\check{5}}T_{3\hat{8}}T_{3\hat{7}}T_{3\hat{6}} \tag{41}$$

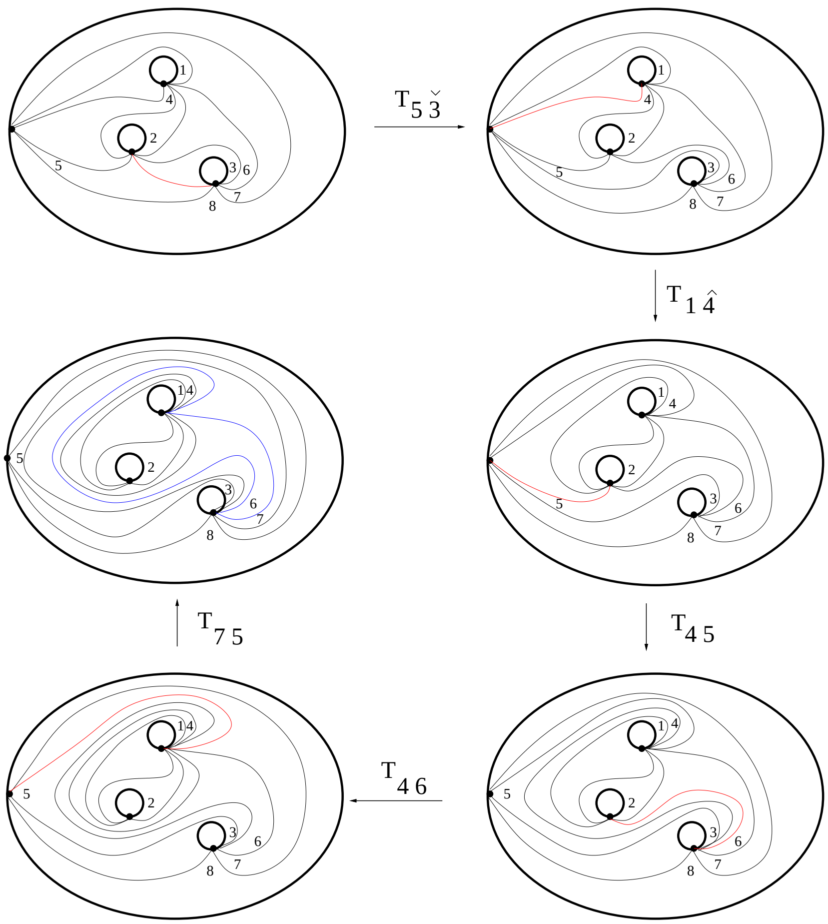
We use further the same recipe for the remaining Dehn twists along boundary components and obtain:

$$\bar{\mathsf{F}}_{a_1} = \bar{\mathsf{F}}(\tau, D_{a_1}\tau) = T_{24}T_{25}T_{2\check{3}}T_{26} \tag{42}$$

$$\bar{\mathsf{F}}_{a_2} = \bar{\mathsf{F}}(\tau, D_{a_2}\tau) = T_{1\hat{4}}T_{1\check{2}}T_{1\check{6}}T_{17} \tag{43}$$

$$\bar{\mathsf{F}}_{a_0} = \bar{\mathsf{F}}(\tau, D_{a_0}\tau) = T_{\check{8}\hat{5}}T_{\check{8}\check{4}}T_{\check{8}\check{1}}T_{\check{8}\check{7}} \tag{44}$$

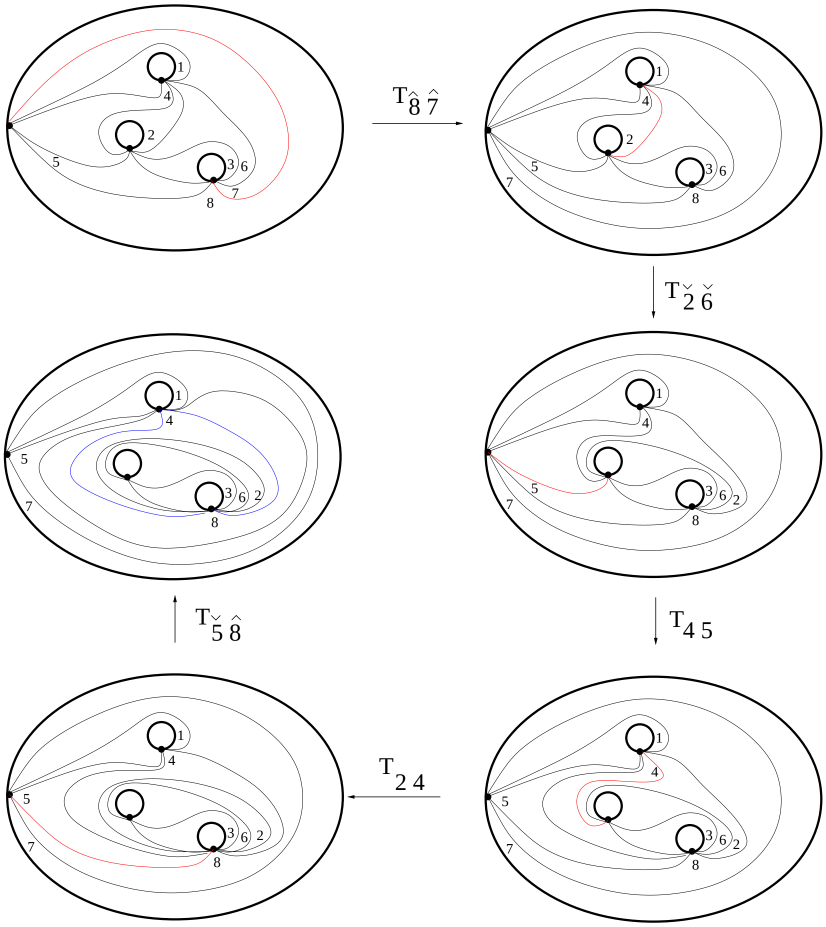
In order to compute $T_{a_{12}}$ we need to transform the triangulation τ into one which intersects a curve isotopic to a_{12} into precisely two points. This can be done as follows:



Therefore we have:

$$\bar{\mathsf{F}}_{a_{12}} = \bar{\mathsf{F}}(\tau, D_{a_{12}}\tau) = Ad(T_{3\check{5}}T_{1\check{4}}T_{45}T_{46}T_{75})(T_{\check{6}7}) \tag{45}$$

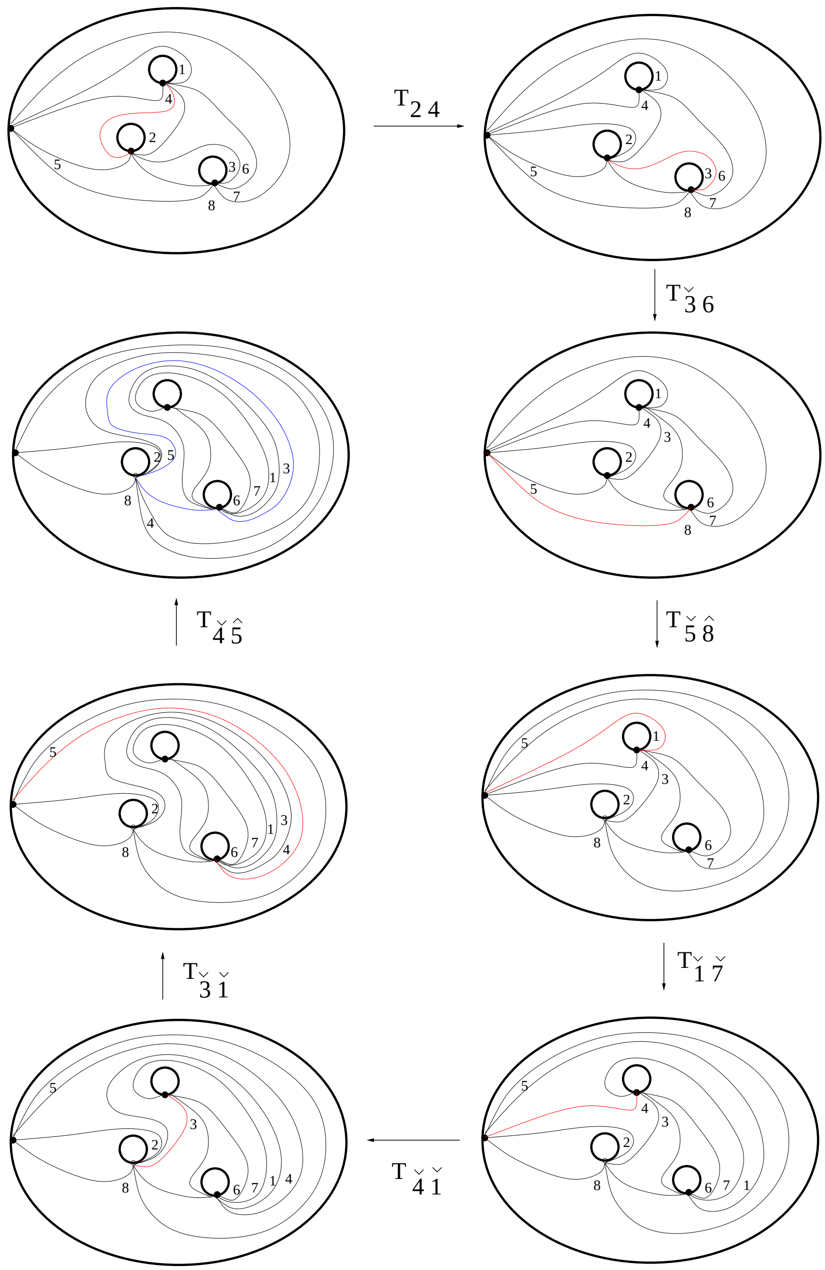
The following sequence of transformations



can be used to compute:

$$\bar{\mathsf{F}}_{a_{13}} = \bar{\mathsf{F}}(\tau, D_{a_{13}}\tau) = Ad(T_{\hat{8}\hat{7}} T_{\check{2}\check{6}} T_{45} T_{24} T_{\check{5}\hat{8}})(T_{\check{4}\check{5}}) \tag{46}$$

Eventually use the transformations



in order to obtain:

$$\bar{\mathsf{F}}_{a_{23}} = \bar{\mathsf{F}}(\tau, D_{a_{23}}\tau) = Ad(T_{\hat4\hat2}T_{\check36}T_{\check5\hat8}T_{\check1\check7}T_{\check4\check1}T_{\check3\check1}T_{\check4\hat5})(T_{3\check5}) \tag{47}$$

The next step is to simplify the expression of the last three Dehn twist, as follows:

$$\begin{aligned}
\bar{\mathsf{F}}_{a_{12}} &= T_{3\check5}T_{1\check4}T_{45}\underline{T_{46}T_{75}}T_{\check67}\underline{\bar T_{75}\bar T_{46}}\bar T_{45}\bar T_{1\check4}\bar T_{3\check5} = T_{3\check5}T_{1\check4}T_{45}T_{75}\underline{T_{46}T_{\check67}\bar T_{46}}\bar T_{75}\bar T_{45}\bar T_{1\check4}\bar T_{3\check5} = \\
&= T_{3\check5}T_{1\check4}T_{45}T_{75}T_{\check67}\underline{\bar T_{47}\bar T_{75}\bar T_{45}}\bar T_{1\check4}\bar T_{3\check5} = T_{3\check5}T_{1\check4}T_{45}\underline{T_{75}T_{\check67}\bar T_{75}}T_{47}\bar T_{1\check4}\bar T_{3\check5} = T_{3\check5}T_{1\check4}T_{45}T_{\check67}T_{\check65}T_{47}\bar T_{1\check4}\bar T_{3\check5} = \\
&= T_{1\check4}T_{\check67}T_{3\check5}T_{45}T_{\check65}\bar T_{3\check5}T_{47}\bar T_{1\check4} = T_{1\check4}T_{\check67}\underline{T_{3\check5}T_{45}\bar T_{3\check5}}\ \underline{T_{3\check5}T_{\check65}\bar T_{3\check5}}T_{47}\bar T_{1\check4} = T_{1\check4}T_{\check67}T_{45}T_{3\check4}T_{\check65}T_{3\hat6}T_{47}\bar T_{1\check4}
\end{aligned}$$

$$\begin{aligned}
\bar{\mathsf{F}}_{a_{13}} &= T_{\hat8\hat7}T_{\check2\check6}T_{45}\underline{T_{24}T_{\check5\hat8}}T_{\check4\check5}\underline{\bar T_{\check5\hat8}\bar T_{24}}\bar T_{45}\bar T_{\check2\check6}\bar T_{\hat8\hat7} = T_{\hat8\hat7}T_{\check2\check6}T_{45}T_{\check5\hat8}\underline{T_{24}T_{\check4\check5}\bar T_{24}}\bar T_{\check5\hat8}\bar T_{45}\bar T_{\check2\check6}\bar T_{\hat8\hat7} = \\
&= T_{\hat8\hat7}T_{\check2\check6}\underline{T_{45}T_{\check5\hat8}T_{\check4\check5}}T_{5\check2}\bar T_{\check5\hat8}\bar T_{45}\bar T_{\check2\check6}\bar T_{\hat8\hat7} = \zeta T_{\hat8\hat7}T_{\check2\check6}T_{\check5\hat8}T_{4\hat8}P_{(45\hat4)}\bar T_{5\check2}\bar T_{\check5\hat8}\bar T_{45}\bar T_{\check2\check6}\bar T_{\hat8\hat7} = \\
&= \zeta T_{\hat8\hat7}T_{\check2\check6}T_{\check5\hat8}\underline{T_{4\hat8}\bar T_{\hat4\check2}\bar T_{4\hat8}}\bar T_{5\hat4}\bar T_{\check2\check6}\bar T_{\hat8\hat7}P_{(45\hat4)} = \zeta T_{\hat8\hat7}\underline{T_{\check2\check6}T_{\check5\hat8}}T_{24}T_{2\hat8}\underline{\bar T_{5\hat4}\bar T_{\check2\check6}}\bar T_{\hat8\hat7}P_{(45\hat4)} = \\
&= \zeta T_{\hat8\hat7}T_{\check5\hat8}T_{\check2\check6}T_{24}T_{2\hat8}\bar T_{\check2\check6}\bar T_{5\hat4}\bar T_{\hat8\hat7}P_{(45\hat4)} = \zeta T_{\hat8\hat7}T_{\check5\hat8}\underline{T_{\check2\check6}T_{24}\bar T_{\check2\check6}}\,\underline{T_{\check2\check6}T_{2\hat8}\bar T_{\check2\check6}}\bar T_{5\hat4}\bar T_{\hat8\hat7}P_{(45\hat4)} = \\
&= \zeta T_{\hat8\hat7}T_{\check5\hat8}T_{24}T_{\hat4\check6}T_{2\hat8}T_{\check8\check6}\bar T_{5\hat4}\bar T_{\hat8\hat7}P_{(45\hat4)}
\end{aligned}$$

$$\begin{aligned}
\bar{\mathsf F}_{a_{23}} &= T_{24}T_{\breve{3}6}T_{\breve{5}\hat{8}}T_{\breve{1}\breve{7}}\underline{T_{\breve{4}\breve{1}}T_{\breve{3}\breve{1}}}T_{\breve{4}\hat{5}}T_{3\breve{5}}\underline{\bar T_{\breve{4}\hat{5}}\bar T_{\breve{3}\breve{1}}}\bar T_{\breve{4}\breve{1}}\bar T_{\breve{1}\breve{7}}\bar T_{\breve{5}\hat{8}}\bar T_{\breve{3}6}\bar T_{24} = \\
&= T_{24}T_{\breve{3}6}T_{\breve{5}\hat{8}}T_{\breve{1}\breve{7}}T_{\breve{4}\breve{1}}T_{\breve{4}\hat{5}}\underline{T_{\breve{3}\breve{1}}T_{3\breve{5}}\bar T_{\breve{3}\breve{1}}}\bar T_{\breve{4}\hat{5}}\bar T_{\breve{4}\breve{1}}\bar T_{\breve{1}\breve{7}}\bar T_{\breve{5}\hat{8}}\bar T_{\breve{3}6}\bar T_{24} = \\
&= T_{24}T_{\breve{3}6}T_{\breve{5}\hat{8}}T_{\breve{1}\breve{7}}T_{\breve{4}\breve{1}}T_{\breve{4}\hat{5}}T_{3\breve{5}}\underline{T_{5\breve{1}}\bar T_{\breve{4}\hat{5}}\bar T_{\breve{4}\breve{1}}}\bar T_{\breve{1}\breve{7}}\bar T_{\breve{5}\hat{8}}\bar T_{\breve{3}6}\bar T_{24} = T_{24}T_{\breve{3}6}T_{\breve{5}\hat{8}}T_{\breve{1}\breve{7}}T_{\breve{4}\breve{1}}\underline{T_{\breve{4}\hat{5}}T_{3\breve{5}}\bar T_{\breve{4}\hat{5}}}T_{5\breve{1}}\bar T_{\breve{1}\breve{7}}\bar T_{\breve{5}\hat{8}}\bar T_{\breve{3}6}\bar T_{24} = \\
&= T_{24}T_{\breve{3}6}T_{\breve{5}\hat{8}}T_{\breve{1}\breve{7}}T_{\breve{4}\breve{1}}T_{\breve{4}\hat{5}}T_{3\breve{5}}\underline{T_{5\breve{1}}\bar T_{\breve{4}\hat{5}}\bar T_{\breve{4}\breve{1}}}\bar T_{\breve{1}\breve{7}}\bar T_{\breve{5}\hat{8}}\bar T_{\breve{3}6}\bar T_{24} = T_{24}T_{\breve{3}6}T_{\breve{5}\hat{8}}T_{\breve{1}\breve{7}}T_{\breve{4}\breve{1}}T_{3\breve{5}}T_{3\hat{4}}T_{5\breve{1}}\bar T_{\breve{1}\breve{7}}\bar T_{\breve{5}\hat{8}}\bar T_{\breve{3}6}\bar T_{24}
\end{aligned}$$

Putting all these together we obtain:

$$\begin{aligned}
\bar{\mathsf F}_{a_{12}}\bar{\mathsf F}_{a_{23}}\bar{\mathsf F}_{a_{13}} &= \zeta T_{1\breve{4}}T_{\breve{6}7}T_{45}T_{3\breve{4}}T_{\breve{6}5}T_{3\hat{6}}T_{47}\bar T_{1\breve{4}}T_{24}T_{\breve{3}6}T_{\breve{5}\hat{8}}T_{\breve{1}\breve{7}}T_{\breve{4}\breve{1}}T_{3\breve{5}}T_{3\hat{4}}T_{5\breve{1}}\bar T_{\breve{1}\breve{7}}\underline{\bar T_{\breve{5}\hat{8}}}\bar T_{\breve{3}6}\underline{\bar T_{24}}\underline{T_{\hat{8}\hat{7}}T_{\breve{5}\hat{8}}}\underline{T_{24}}T_{\hat{4}\breve{6}}T_{2\hat{8}}T_{\breve{8}\breve{6}}\times \\
&\times \bar T_{5\hat{4}}\bar T_{\hat{8}\hat{7}}P_{(45\hat{4})} = \zeta T_{1\breve{4}}T_{\breve{6}7}T_{45}T_{3\breve{4}}T_{\breve{6}5}T_{3\hat{6}}T_{47}\bar T_{1\breve{4}}T_{24}T_{\breve{3}6}T_{\breve{5}\hat{8}}T_{\breve{1}\breve{7}}T_{\breve{4}\breve{1}}T_{3\breve{5}}T_{3\hat{4}}\underline{T_{5\breve{1}}\bar T_{\breve{1}\breve{7}}}\bar T_{\breve{3}6}\underline{T_{\breve{5}\hat{7}}}T_{\hat{8}\hat{7}}T_{\hat{4}\breve{6}}T_{2\hat{8}}T_{\breve{8}\breve{6}}\bar T_{5\hat{4}}\times \\
&\times \bar T_{\hat{8}\hat{7}}P_{(45\hat{4})} = \zeta^2 T_{1\breve{4}}T_{\breve{6}7}T_{45}T_{3\breve{4}}T_{\breve{6}5}T_{3\hat{6}}T_{47}\bar T_{1\breve{4}}T_{24}T_{\breve{3}6}T_{\breve{5}\hat{8}}\underline{T_{\breve{1}\breve{7}}T_{\breve{4}\breve{1}}}T_{3\breve{5}}T_{3\hat{4}}\underline{\bar T_{\breve{1}\breve{7}}}T_{5\breve{1}}\underline{P_{(5\breve{7}\hat{5})}}\bar T_{\breve{3}6}T_{\hat{8}\hat{7}}T_{\hat{4}\breve{6}}T_{2\hat{8}}T_{\breve{8}\breve{6}}\bar T_{5\hat{4}}\times \\
&\times \bar T_{\hat{8}\hat{7}}P_{(45\hat{4})} = \zeta^2 T_{1\breve{4}}T_{\breve{6}7}T_{45}T_{3\breve{4}}T_{\breve{6}5}T_{3\hat{6}}T_{47}\underline{\bar T_{1\breve{4}}T_{24}}T_{\breve{3}6}T_{\breve{5}\hat{8}}\underline{T_{\breve{4}\breve{1}}}T_{\breve{4}\breve{7}}T_{3\breve{5}}T_{3\hat{4}}T_{5\breve{1}}\bar T_{\breve{3}6}T_{\hat{4}\breve{6}}T_{\hat{8}5}T_{2\hat{8}}T_{\breve{8}\breve{6}}\bar T_{\hat{8}5}\bar T_{\breve{7}\hat{4}}\times \\
&\times P_{(5\breve{7}\hat{5})}P_{(45\hat{4})} = \zeta^2 T_{1\breve{4}}T_{\breve{6}7}T_{45}T_{3\breve{4}}T_{\breve{6}5}T_{3\hat{6}}\underline{T_{47}}T_{1\breve{2}}\underline{T_{24}}T_{\breve{3}6}T_{\breve{5}\hat{8}}\underline{T_{\breve{4}\breve{7}}}T_{3\breve{5}}T_{3\hat{4}}T_{5\breve{1}}\bar T_{\breve{3}6}T_{\hat{4}\breve{6}}T_{\hat{8}5}T_{2\hat{8}}T_{\breve{8}\breve{6}}\bar T_{\hat{8}5}\bar T_{\breve{7}\hat{4}}\times \\
&\times P_{(5\breve{7}\hat{5})}P_{(45\hat{4})} = \zeta^3 T_{1\breve{4}}\underline{T_{\breve{6}7}T_{45}T_{3\breve{4}}T_{\breve{6}5}T_{3\hat{6}}}\ \underline{T_{1\breve{2}}}T_{24}T_{27}P_{(47\hat{4})}T_{\breve{3}6}T_{\breve{5}\hat{8}}T_{3\breve{5}}T_{3\hat{4}}T_{5\breve{1}}\bar T_{\breve{3}6}T_{\hat{4}\breve{6}}T_{\hat{8}5}T_{2\hat{8}}T_{\breve{8}\breve{6}}\bar T_{\hat{8}5}\bar T_{\breve{7}\hat{4}}\times \\
&\times P_{(5\breve{7}\hat{5})}P_{(45\hat{4})} = \zeta^3 T_{1\breve{4}}T_{1\breve{2}}\underline{T_{1\breve{6}}T_{17}\bar T_{17}\bar T_{1\breve{6}}}T_{\breve{6}7}\underline{T_{24}\bar T_{24}}T_{45}T_{3\breve{4}}\underline{T_{\breve{6}5}T_{3\hat{6}}}\ \underline{T_{24}}T_{27}P_{(47\hat{4})}T_{\breve{3}6}T_{\breve{5}\hat{8}}T_{3\breve{5}}T_{3\hat{4}}T_{5\breve{1}}\bar T_{\breve{3}6}T_{\hat{4}\breve{6}}T_{\hat{8}5}T_{2\hat{8}}\times \\
&\times T_{\breve{8}\breve{6}}\bar T_{\hat{8}5}\bar T_{\breve{7}\hat{4}}P_{(5\breve{7}\hat{5})}P_{(45\hat{4})} = \zeta^3\bar{\mathsf F}_{a_2}\underline{\bar T_{17}\bar T_{\breve{6}7}\bar T_{1\breve{6}}}T_{24}\underline{\bar T_{24}T_{45}T_{24}}\ \underline{\bar T_{24}T_{3\breve{4}}T_{24}}T_{\breve{6}5}\underline{T_{3\hat{6}}T_{27}}\ \underline{P_{(47\hat{4})}T_{\breve{3}6}}T_{\breve{5}\hat{8}}T_{3\breve{5}}T_{3\hat{4}}T_{5\breve{1}}\bar T_{\breve{3}6}T_{\hat{4}\breve{6}}\times \\
&\times T_{\hat{8}5}T_{2\hat{8}}T_{\breve{8}\breve{6}}\bar T_{\hat{8}5}\bar T_{\breve{7}\hat{4}}P_{(5\breve{7}\hat{5})}P_{(45\hat{4})} = \zeta^3\bar{\mathsf F}_{a_2}T_{\breve{6}7}\bar T_{1\breve{6}}T_{24}T_{25}\underline{T_{45}T_{2\breve{3}}}T_{4\breve{3}}\underline{T_{\breve{6}5}T_{27}}\underline{T_{3\hat{6}}T_{\breve{3}6}}P_{(47\hat{4})}T_{\breve{5}\hat{8}}T_{3\breve{5}}T_{3\hat{4}}T_{5\breve{1}}\bar T_{\breve{3}6}T_{\hat{4}\breve{6}}\times \\
&\times T_{\hat{8}5}T_{2\hat{8}}T_{\breve{8}\breve{6}}\bar T_{\hat{8}5}\bar T_{\breve{7}\hat{4}}P_{(5\breve{7}\hat{5})}P_{(45\hat{4})} = \zeta^4\bar{\mathsf F}_{a_2}\underline{T_{24}T_{25}T_{2\breve{3}}T_{26}}\ \underline{\bar T_{26}T_{\breve{6}7}}\bar T_{1\breve{6}}T_{45}T_{4\breve{3}}\underline{T_{27}}T_{\breve{6}5}P_{(\breve{6}\breve{3}6)}P_{(47\hat{4})}T_{\breve{5}\hat{8}}T_{3\breve{5}}T_{3\hat{4}}T_{5\breve{1}}\bar T_{\breve{3}6}T_{\hat{4}\breve{6}}\times \\
&\times T_{\hat{8}5}T_{2\hat{8}}T_{\breve{8}\breve{6}}\bar T_{\hat{8}5}\bar T_{\breve{7}\hat{4}}P_{(5\breve{7}\hat{5})}P_{(45\hat{4})} = \zeta^4\bar{\mathsf F}_{a_2}\bar{\mathsf F}_{a_1}T_{\breve{6}7}\bar T_{26}\bar T_{1\breve{6}}T_{45}T_{4\breve{3}}\underline{T_{\breve{6}5}T_{\breve{5}\hat{8}}T_{\hat{6}\breve{5}}}\ \underline{T_{6\hat{7}}}T_{5\breve{1}}\underline{\bar T_{63}T_{\hat{7}\breve{3}}}T_{\hat{8}5}T_{2\hat{8}}T_{\breve{8}\breve{3}}\bar T_{\hat{8}5}\bar T_{4\hat{7}}\times \\
&\times P_{(\breve{6}\breve{3}6)}P_{(47\hat{4})}P_{(5\breve{7}\hat{5})}P_{(45\hat{4})} = \zeta^6\bar{\mathsf F}_{a_2}\bar{\mathsf F}_{a_1}T_{\breve{6}7}\bar T_{26}\underline{\bar T_{1\breve{6}}}T_{45}T_{4\breve{3}}T_{\breve{5}\hat{8}}\underline{T_{\breve{6}\hat{8}}T_{6\breve{1}}}\bar T_{\hat{5}3}T_{\breve{5}\hat{7}}T_{\hat{8}6}T_{2\hat{8}}T_{\breve{8}\breve{7}}\bar T_{\hat{8}6}\bar T_{4\bar{3}}\times \\
&\times P_{(\breve{6}56)}P_{(\hat{3}\hat{7}\breve{3})}P_{(\breve{6}\breve{3}6)}P_{(47\hat{4})}P_{(5\breve{7}\hat{5})}P_{(45\hat{4})} = \zeta^6\bar{\mathsf F}_{a_2}\bar{\mathsf F}_{a_1}T_{\breve{6}7}\bar T_{26}T_{45}T_{4\breve{3}}T_{\breve{5}\hat{8}}T_{1\hat{8}}\underline{T_{\breve{6}\hat{8}}}\bar T_{\hat{5}3}T_{\breve{5}\hat{7}}\underline{T_{\hat{8}6}}T_{2\hat{8}}T_{\breve{8}\breve{7}}\bar T_{\hat{8}6}\bar T_{4\bar{3}}\times \\
&\times P_{(\breve{6}56)}P_{(\hat{3}\hat{7}\breve{3})}P_{(\breve{6}\breve{3}6)}P_{(47\hat{4})}P_{(5\breve{7}\hat{5})}P_{(45\hat{4})} = \zeta^7\bar{\mathsf F}_{a_2}\bar{\mathsf F}_{a_1}T_{\breve{6}7}\bar T_{26}T_{45}T_{4\breve{3}}T_{\breve{5}\hat{8}}T_{1\hat{8}}\bar T_{\hat{5}3}T_{\breve{5}\hat{7}}P_{(\breve{6}\hat{8}6)}T_{\hat{8}}T_{\breve{8}\breve{7}}\bar T_{\hat{8}6}\bar T_{4\bar{3}}\times \\
&\times P_{(\breve{6}56)}P_{(\hat{3}\hat{7}\breve{3})}P_{(\breve{6}\breve{3}6)}P_{(47\hat{4})}P_{(5\breve{7}\hat{5})}P_{(45\hat{4})} = \zeta^7\bar{\mathsf F}_{a_2}\bar{\mathsf F}_{a_1}\underline{T_{\breve{6}7}}\ \underline{\bar T_{26}}T_{45}T_{4\breve{3}}T_{\breve{5}\hat{8}}T_{1\hat{8}}\bar T_{\hat{5}3}\underline{T_{\breve{5}\hat{7}}}\ \underline{T_{26}}\ \underline{T_{\hat{6}\breve{7}}}\bar T_{6\breve{8}}\bar T_{4\bar{3}}\times \\
&\times P_{(\breve{6}\hat{8}6)}P_{(\breve{6}56)}P_{(\hat{3}\hat{7}\breve{3})}P_{(\breve{6}\breve{3}6)}P_{(47\hat{4})}P_{(5\breve{7}\hat{5})}P_{(45\hat{4})} = \zeta^8\bar{\mathsf F}_{a_2}\bar{\mathsf F}_{a_1}T_{45}T_{4\breve{3}}T_{\breve{5}\hat{8}}T_{1\hat{8}}\bar T_{\hat{5}3}T_{\breve{5}\hat{7}}T_{\breve{6}\hat{5}}P_{(\breve{6}76)}\bar T_{6\breve{8}}\bar T_{4\bar{3}}\times \\
&\times P_{(\breve{6}\hat{8}6)}P_{(\breve{6}56)}P_{(\hat{3}\hat{7}\breve{3})}P_{(\breve{6}\breve{3}6)}P_{(47\hat{4})}P_{(5\breve{7}\hat{5})}P_{(45\hat{4})} = \zeta^8\bar{\mathsf F}_{a_2}\bar{\mathsf F}_{a_1}\bar{\mathsf F}_{a_0}\bar T_{\breve{8}\breve{7}}\bar T_{\breve{8}\breve{1}}\bar T_{\breve{8}\breve{4}}\underline{\bar T_{\breve{8}\hat{5}}T_{45}}T_{4\breve{3}}\underline{T_{\breve{5}\hat{8}}}T_{1\hat{8}}\bar T_{\hat{5}3}T_{\breve{5}\hat{7}}T_{\breve{6}\hat{5}}\bar T_{\hat{7}\breve{8}}\bar T_{4\bar{3}}\times \\
&\times P_{(\breve{6}76)}P_{(\breve{6}\hat{8}6)}P_{(\breve{6}56)}P_{(\hat{3}\hat{7}\breve{3})}P_{(\breve{6}\breve{3}6)}P_{(47\hat{4})}P_{(5\breve{7}\hat{5})}P_{(45\hat{4})} = \zeta^8\bar{\mathsf F}_{a_2}\bar{\mathsf F}_{a_1}\bar{\mathsf F}_{a_0}\bar T_{\breve{8}\breve{7}}\underline{\bar T_{\breve{8}\breve{1}}}\ \underline{\bar T_{\breve{8}\breve{4}}}\ \underline{T_{4\hat{8}}}T_{45}T_{4\breve{3}}\underline{T_{1\hat{8}}}\bar T_{\hat{5}3}T_{\breve{5}\hat{7}}T_{\breve{6}\hat{5}}\bar T_{\hat{7}\breve{8}}\bar T_{4\bar{3}}\times \\
&\times P_{(\breve{6}76)}P_{(\breve{6}\hat{8}6)}P_{(\breve{6}56)}P_{(\hat{3}\hat{7}\breve{3})}P_{(\breve{6}\breve{3}6)}P_{(47\hat{4})}P_{(5\breve{7}\hat{5})}P_{(45\hat{4})} = \zeta^8\bar{\mathsf F}_{a_2}\bar{\mathsf F}_{a_1}\bar{\mathsf F}_{a_0}\underline{\bar T_{\breve{8}\breve{7}}}T_{45}T_{4\breve{3}}\bar T_{\hat{5}3}\underline{T_{\breve{5}\hat{7}}}T_{\breve{6}\hat{5}}\underline{\bar T_{\hat{7}\breve{8}}}\bar T_{4\bar{3}}\times \\
&\times P_{(\breve{6}76)}P_{(\breve{6}\hat{8}6)}P_{(\breve{6}56)}P_{(\hat{3}\hat{7}\breve{3})}P_{(\breve{6}\breve{3}6)}P_{(47\hat{4})}P_{(5\breve{7}\hat{5})}P_{(45\hat{4})}
\end{aligned}$$

Use now the identity:

$$\bar T_{\breve{8}\breve{7}}T_{\breve{5}\hat{7}}\bar T_{\hat{7}\breve{8}} = T_{\breve{5}\hat{8}}T_{\breve{5}\hat{7}}\bar T_{\breve{8}\breve{7}}\bar T_{\hat{7}\breve{8}} = \zeta^{-1}T_{\breve{5}\hat{8}}T_{\breve{5}\hat{7}}P_{(\breve{7}\breve{8}\hat{7})}$$

and introduce above to find that:

$$\begin{aligned}
\bar{\mathsf F}_{a_{12}}\bar{\mathsf F}_{a_{23}}\bar{\mathsf F}_{a_{13}} &= \zeta^7\bar{\mathsf F}_{a_2}\bar{\mathsf F}_{a_1}\bar{\mathsf F}_{a_0}T_{45}T_{4\breve{3}}\bar T_{\hat{5}3}T_{\breve{5}\hat{8}}T_{\breve{5}\hat{7}}T_{\breve{6}\hat{5}}\bar T_{4\bar{3}}\times \\
&\times P_{(\breve{7}\breve{8}\hat{7})}P_{(\breve{6}76)}P_{(\breve{6}\hat{8}6)}P_{(\breve{6}56)}P_{(\hat{3}\hat{7}\breve{3})}P_{(\breve{6}\breve{3}6)}P_{(47\hat{4})}P_{(5\breve{7}\hat{5})}P_{(45\hat{4})} = \zeta^7\bar{\mathsf F}_{a_2}\bar{\mathsf F}_{a_1}\bar{\mathsf F}_{a_0}\bar{\mathsf F}_{a_3}\bar T_{3\hat{6}}\bar T_{3\hat{7}}\bar T_{3\hat{8}}\underline{\bar T_{3\breve{5}}T_{45}T_{4\breve{3}}}\bar T_{\hat{5}3}T_{\breve{5}\hat{8}}T_{\breve{5}\hat{7}}T_{\breve{6}\hat{5}}\bar T_{4\bar{3}}\times \\
&\times P_{(\breve{7}\breve{8}\hat{7})}P_{(\breve{6}76)}P_{(\breve{6}\hat{8}6)}P_{(\breve{6}56)}P_{(\hat{3}\hat{7}\breve{3})}P_{(\breve{6}\breve{3}6)}P_{(47\hat{4})}P_{(5\breve{7}\hat{5})}P_{(45\hat{4})} = \zeta^7\bar{\mathsf F}_{a_2}\bar{\mathsf F}_{a_1}\bar{\mathsf F}_{a_0}\bar{\mathsf F}_{a_3}\bar T_{3\hat{6}}\bar T_{3\hat{7}}\bar T_{3\hat{8}}T_{45}\underline{\bar T_{3\breve{5}}\bar T_{\hat{5}3}}T_{\breve{5}\hat{8}}T_{\breve{5}\hat{7}}T_{\breve{6}\hat{5}}\bar T_{4\bar{3}}\times \\
&\times P_{(\breve{7}\breve{8}\hat{7})}P_{(\breve{6}76)}P_{(\breve{6}\hat{8}6)}P_{(\breve{6}56)}P_{(\hat{3}\hat{7}\breve{3})}P_{(\breve{6}\breve{3}6)}P_{(47\hat{4})}P_{(5\breve{7}\hat{5})}P_{(45\hat{4})} = \zeta^6\bar{\mathsf F}_{a_2}\bar{\mathsf F}_{a_1}\bar{\mathsf F}_{a_0}\bar{\mathsf F}_{a_3}\bar T_{3\hat{6}}\bar T_{3\hat{7}}\bar T_{3\hat{8}}T_{45}P_{(\breve{5}3\hat{5})}T_{\breve{5}\hat{8}}T_{\breve{5}\hat{7}}T_{\breve{6}\hat{5}}\bar T_{4\bar{3}}\times \\
&\times P_{(\breve{7}\breve{8}\hat{7})}P_{(\breve{6}76)}P_{(\breve{6}\hat{8}6)}P_{(\breve{6}56)}P_{(\hat{3}\hat{7}\breve{3})}P_{(\breve{6}\breve{3}6)}P_{(47\hat{4})}P_{(5\breve{7}\hat{5})}P_{(45\hat{4})} = \zeta^6\bar{\mathsf F}_{a_2}\bar{\mathsf F}_{a_1}\bar{\mathsf F}_{a_0}\bar{\mathsf F}_{a_3}\underline{\bar T_{3\hat{6}}}\ \underline{\bar T_{3\hat{7}}}\ \underline{\bar T_{3\hat{8}}}\ \underline{T_{45}}\ \underline{T_{3\hat{8}}}\ \underline{T_{3\hat{7}}T_{\breve{6}\breve{3}}}\ \underline{\bar T_{45}}\times \\
&\times P_{(\breve{5}3\hat{5})}P_{(\breve{7}\breve{8}\hat{7})}P_{(\breve{6}76)}P_{(\breve{6}\hat{8}6)}P_{(\breve{6}56)}P_{(\hat{3}\hat{7}\breve{3})}P_{(\breve{6}\breve{3}6)}P_{(47\hat{4})}P_{(5\breve{7}\hat{5})}P_{(45\hat{4})} = \zeta^6\bar{\mathsf F}_{a_2}\bar{\mathsf F}_{a_1}\bar{\mathsf F}_{a_0}\bar{\mathsf F}_{a_3}
\end{aligned}$$

Thus the lift of the lantern relation is ζ^6. Therefore we have to renormalize each right Dehn twist by taking $\widetilde{D_\alpha} = \zeta^{-6}\mathsf F_\alpha$, as claimed. $\square$

We will suppose henceforth that the lifts are normalized.

Lemma 2.6. *Let a, b, c, e, f be the five curves appearing in the chain relation $(D_aD_bD_c)^4 = D_eD_f$ on an embedded 2-holed torus sitting inside $\Sigma_{g,r}^s$. If $s \geq 2$, then the lifts of Dehn twists in $\widetilde{\Gamma}_{g,r}^s$ satisfy the relation*

$$(\widetilde{D}_a\widetilde{D}_b\widetilde{D}_c)^4 = \zeta^{-72}\widetilde{D}_e\widetilde{D}_f$$

Proof. If $s \geq 2$ and $g \geq 2$, then there is an embedding $\Sigma^2_{2,1} \subset \Sigma^s_{g,r}$.

We consider a surface S homeomorphic to $\Sigma^2_{1,2}$, i.e. a torus with two holes and two punctures drawn in the left picture of Figure 5 where the opposite sides of the rectangle are identified. Notice that the two punctures are located on the two boundary components. The central picture of Figure 5 specifies five simple closed

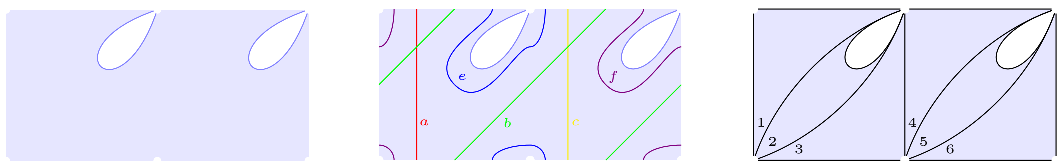


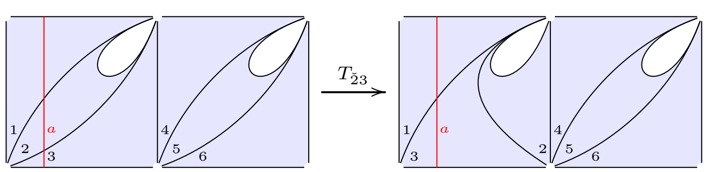
Figure 5

curves a, b, c, e, f in S, the Dehn twists along which enter the chain relation.

We also choose a particular decorated ideal triangulation τ of S given by the right picture of Figure 5, where the ideal arcs are drawn in black and the positions of the numbers in ideal triangles correspond to the marked corners. Notice that our choice is manifestly symmetric with respect to the exchange of the left and the right halves of the rectangle accompanied with relabeling $(1, 2, 3) \leftrightarrow (4, 5, 6)$. This symmetry will be useful for reducing the amount of calculations in deriving the quantum realizations of the Dehn twists.

The basic procedure in deriving the quantum realization of the Dehn twist D_α along a given simple closed curve α is to use a specific decorated ideal triangulation where the contour α intersects only two ideal arcs, so that the annular neighborhood of α is given by only two ideal triangles. With respect to such (decorated) ideal triangulation the quantum operator realizing D_α is given by a single T-operator. Let us work out this procedure in the case of the curves a, b, c, e, f.

For any simple closed curve α, we denote $\bar{\mathsf{F}}_\alpha = \widetilde{D}_\alpha^{-1} \simeq \mathsf{F}(D_\alpha \tau, \tau)$. To derive the operator representing the Dehn twist D_a, we apply the following change of triangulation:



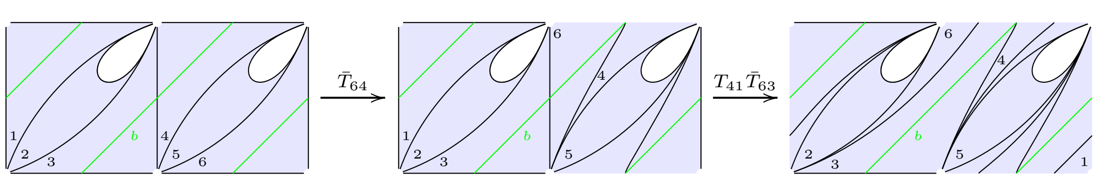
where the operator above the arrow realizes the corresponding element of the groupoid of decorated ideal triangulations within the quantum Teichmüller theory. Thus,

$$\zeta^{-6}\bar{\mathsf{F}}_a = \mathrm{Ad}(T_{\check{2}3})(T_{1\hat{3}}) = T_{\check{2}3}T_{1\hat{3}}\bar{T}_{\check{2}3} = T_{1\hat{3}}T_{1\hat{2}},$$

where in the last equality, we have applied once the Pentagon relation, and we use the notation $\bar{T} = T^{-1}$. Here, we use the normalization where the braid-type and the lantern relations are satisfied without projective factors. By the above mentioned left-right symmetry $(1, 2, 3) \leftrightarrow (4, 5, 6)$, we immediately get the quantum realization of the Dehn twist D_c:

$$\zeta^{-6}\bar{\mathsf{F}}_c = T_{4\hat{6}}T_{4\hat{5}}.$$

To calculate the quantum realization of D_b we use a two-step chain of transformations of τ:

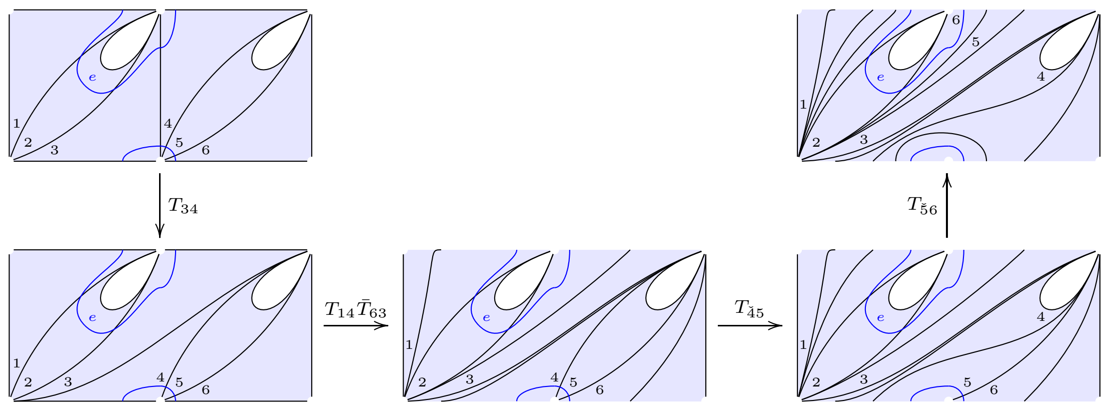


Thus, we have the following sequence of equalities:

$$\zeta^{-6}\bar{\mathsf{F}}_b = \mathrm{Ad}(\bar{T}_{64}T_{41}\bar{T}_{63})(T_{34}) = \bar{T}_{64}T_{41}\underline{\bar{T}_{63}T_{34}T_{63}}\bar{T}_{41}T_{64}$$
$$= \underline{\bar{T}_{64}T_{41}T_{64}}T_{34}\bar{T}_{41}T_{64} = T_{61}\underline{T_{41}T_{34}\bar{T}_{41}}T_{64} = T_{61}T_{34}T_{31}T_{64},$$

where in each step the underlined fragment is transformed by using the Pentagon relation.

To calculate the realization of D_e, we consider the following sequence of ideal triangulations:



Thus, we have

$$
\begin{aligned}
\zeta^{-6}\bar{\mathsf{F}}_e &= \mathrm{Ad}(T_{34}T_{14}\bar{T}_{63}T_{\check{4}5}T_{\check{5}6})(T_{\check{2}\hat{6}}) = T_{34}T_{14}\bar{T}_{63}T_{\check{4}5}\underline{T_{\hat{6}\hat{5}}T_{\check{2}\hat{6}}\bar{T}_{\hat{6}\hat{5}}}\bar{T}_{\check{4}5}T_{63}\bar{T}_{14}\bar{T}_{34}\\
&= T_{34}T_{14}\bar{T}_{63}\underline{T_{\hat{5}\hat{4}}}T_{\check{2}\hat{6}}\underline{T_{\check{2}\hat{5}}\bar{T}_{\hat{5}\hat{4}}}T_{63}\bar{T}_{14}\bar{T}_{34} = T_{34}\underline{T_{\hat{4}\check{1}}}\bar{T}_{63}T_{\check{2}\hat{6}}T_{\check{2}\hat{5}}\underline{T_{\check{2}\hat{4}}}T_{63}\underline{\bar{T}_{\hat{4}\check{1}}}\bar{T}_{34}\\
&= T_{34}\underline{\bar{T}_{\hat{3}\check{6}}T_{\check{6}\hat{2}}}T_{\check{2}\hat{5}}T_{\check{2}\hat{4}}T_{\check{2}\check{1}}\underline{T_{\hat{3}\check{6}}}\bar{T}_{34} = \underline{T_{34}T_{\check{2}3}}T_{\check{6}\hat{2}}T_{\check{2}\hat{5}}T_{\check{2}\hat{4}}T_{\check{2}\check{1}}\bar{T}_{34}\\
&= T_{\check{2}3}T_{\check{2}4}\underline{T_{\hat{4}\check{3}}}T_{\check{6}\hat{2}}T_{\check{2}\hat{5}}\underline{T_{\check{2}\hat{4}}}T_{\check{2}\check{1}}\underline{\bar{T}_{\hat{4}\check{3}}} = T_{\check{2}3}T_{\check{2}4}T_{\check{2}\hat{6}}T_{\check{2}\hat{5}}T_{\check{2}\hat{4}}T_{\check{2}\check{3}}T_{\check{2}\check{1}},
\end{aligned}
$$

where, as before, in each step the underlined fragment is transformed by applying the Pentagon relation. Again, using the symmetry $(1,2,3) \leftrightarrow (4,5,6)$, we also have

$$\zeta^{-6}\bar{\mathsf{F}}_f = T_{\check{5}6}T_{\check{5}1}T_{\check{5}\hat{3}}T_{\check{5}\hat{2}}T_{\check{5}\hat{1}}T_{\check{5}\check{6}}T_{\check{5}\check{4}}.$$

In order to check the Chain relation, we first calculate the following product:

$$\zeta^{-18}\bar{\mathsf{F}}_c\bar{\mathsf{F}}_b\bar{\mathsf{F}}_a = T_{4\hat{6}}T_{4\hat{5}}T_{61}T_{34}\underline{T_{31}}T_{64}\underline{T_{1\hat{3}}}T_{1\hat{2}} = T_{4\hat{6}}T_{4\hat{5}}T_{61}T_{34}T_{64}\zeta P_{(31\hat{3})}T_{1\hat{2}} = \zeta T_{4\hat{6}}T_{4\hat{5}}T_{61}T_{34}T_{64}T_{\hat{3}\hat{2}}P_{(31\hat{3})},$$

where we have applied the Inversion relation to the underlined fragment. Next, we calculate

$$
\begin{aligned}
\zeta^{-36}(\bar{\mathsf{F}}_c\bar{\mathsf{F}}_b\bar{\mathsf{F}}_a)^2 &= \zeta^2 T_{4\hat{6}}T_{4\hat{5}}T_{61}T_{34}T_{64}T_{\hat{3}\hat{2}}\underline{P_{(31\hat{3})}}T_{4\hat{6}}T_{4\hat{5}}\underline{T_{61}T_{34}}T_{64}\underline{T_{\hat{3}\hat{2}}P_{(31\hat{3})}}\\
&= \zeta^2 T_{4\hat{6}}T_{4\hat{5}}T_{61}T_{34}\underline{T_{64}}T_{\hat{3}\hat{2}}\underline{T_{4\hat{6}}T_{4\hat{5}}T_{6\hat{3}}T_{14}T_{64}}T_{\hat{1}\hat{2}}P_{(3\hat{3})(1\hat{1})}\\
&= \zeta^3 T_{4\hat{6}}T_{4\hat{5}}\underline{T_{61}T_{34}T_{\hat{3}\hat{2}}T_{\hat{6}\hat{5}}}T_{4\hat{3}}T_{1\hat{6}}T_{4\hat{6}}T_{\hat{1}\hat{2}}P_{(64\hat{6})}P_{(3\hat{3})(1\hat{1})}\\
&= \zeta^3 \underline{T_{4\hat{6}}T_{4\hat{5}}T_{\hat{6}\hat{5}}}T_{\check{5}1}\underline{T_{61}}T_{\hat{3}\hat{2}}T_{\check{2}4}\underline{T_{34}T_{4\hat{3}}T_{1\hat{6}}}T_{4\hat{6}}T_{\hat{1}\hat{2}}P_{(64\hat{6})}P_{(3\hat{3})(1\hat{1})}\\
&= \zeta^5 T_{\hat{6}\hat{5}}T_{4\hat{6}}T_{\check{5}1}T_{\hat{3}\hat{2}}T_{\check{2}4}\underline{P_{(61\hat{6})}P_{(34\hat{3})}T_{4\hat{6}}T_{\hat{1}\hat{2}}P_{(64\hat{6})}P_{(3\hat{3})(1\hat{1})}} = \zeta^5 T_{\hat{6}\hat{5}}T_{4\hat{6}}T_{\check{5}1}T_{\check{2}3}T_{\check{2}4}T_{\hat{3}\hat{1}}T_{\check{6}\hat{2}}P_{(1\check{6}3\hat{4}\check{1})},
\end{aligned}
$$

where each equality is obtained by transforming the underlined fragment by applying the Pentagon relation (twice in the forth and once in the fifth equalities), the Inversion relation (once in the third and twice in the fifth equalities), and the extended symmetric group action (in the second, the third, and the sixth equalities). Finally, taking the square of the obtained identity, we have

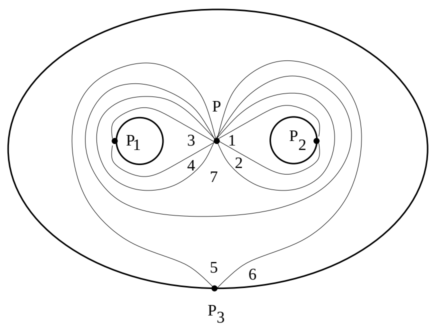
$$
\begin{aligned}
\zeta^{-72}(\bar{\mathsf{F}}_c\bar{\mathsf{F}}_b\bar{\mathsf{F}}_a)^4 &= \zeta^{10} T_{\hat{6}\hat{5}}T_{4\hat{6}}T_{\check{5}1}T_{\check{2}3}T_{\check{2}4}T_{\hat{3}\hat{1}}T_{\check{6}\hat{2}}\underline{P_{(1\check{6}3\hat{4}\check{1})}T_{\hat{6}\hat{5}}T_{4\hat{6}}T_{\check{5}1}T_{\check{2}3}T_{\check{2}4}T_{\hat{3}\hat{1}}T_{\check{6}\hat{2}}P_{(1\check{6}3\hat{4}\check{1})}}\\
&= \zeta^{10} T_{\check{5}6}T_{4\hat{6}}T_{\check{5}1}T_{\check{2}3}T_{\check{2}4}\underline{T_{\hat{3}\hat{1}}}T_{\check{6}\hat{2}}\underline{T_{\check{5}\hat{3}}}T_{\hat{1}\check{3}}T_{\check{5}\check{6}}T_{\check{2}\hat{4}}T_{\check{2}\hat{1}}T_{\check{4}6}T_{3\hat{2}}P_{(13\check{1})}P_{(46\check{4})}\\
&= \zeta^{10} T_{\check{5}6}T_{4\hat{6}}T_{\check{5}1}T_{\check{2}3}T_{\check{2}4}T_{\check{6}\hat{2}}T_{\check{5}\hat{3}}T_{\check{5}\hat{1}}\underline{T_{\hat{3}\hat{1}}T_{\hat{1}\check{3}}}T_{\check{5}\check{6}}T_{\check{2}\hat{4}}\underline{T_{\check{2}\hat{1}}}T_{\check{4}6}\underline{T_{3\hat{2}}P_{(13\check{1})}}P_{(46\check{4})}\\
&= \zeta^{11} T_{\check{5}6}\underline{T_{4\hat{6}}}T_{\check{5}1}T_{\check{2}3}\underline{T_{\check{2}4}}T_{\check{6}\hat{2}}T_{\check{5}\hat{3}}T_{\check{5}\hat{1}}T_{\check{5}\check{6}}T_{\check{2}\hat{4}}T_{\check{2}\check{3}}T_{\check{4}6}T_{1\hat{2}}P_{(46\check{4})}\\
&= \zeta^{11} T_{\check{5}6}T_{\check{5}1}T_{\check{2}3}T_{\check{2}4}T_{\check{2}\hat{6}}\underline{T_{4\hat{6}}T_{\check{6}\hat{2}}}T_{\check{5}\hat{3}}T_{\check{5}\hat{1}}T_{\check{5}\check{6}}\underline{T_{\check{2}\hat{4}}}T_{\check{4}6}T_{\check{2}\check{3}}T_{1\hat{2}}P_{(46\check{4})}\\
&= \zeta^{11} T_{\check{5}6}T_{\check{5}1}T_{\check{2}3}T_{\check{2}4}T_{\check{2}\hat{6}}T_{\check{5}\hat{3}}T_{\check{5}\hat{1}}T_{\check{2}\hat{4}}\underline{T_{4\hat{6}}T_{\check{5}\check{6}}}T_{\check{4}6}T_{\check{2}\check{3}}T_{1\hat{2}}P_{(46\check{4})}\\
&= \zeta^{11} T_{\check{5}6}T_{\check{5}1}T_{\check{2}3}T_{\check{2}4}T_{\check{2}\hat{6}}T_{\check{5}\hat{3}}T_{\check{5}\hat{1}}T_{\check{2}\hat{4}}T_{\check{5}6}T_{\check{5}\check{4}}\underline{T_{4\hat{6}}T_{\check{4}6}}T_{\check{2}\check{3}}T_{1\hat{2}}\underline{P_{(46\check{4})}}\\
&= \zeta^{12} T_{\check{5}6}T_{\check{5}1}\underline{T_{\check{2}3}}T_{\check{2}4}\underline{T_{\check{2}\hat{6}}T_{\check{5}\hat{3}}}T_{\check{5}\hat{1}}\underline{T_{\check{2}\hat{4}}T_{\check{5}\check{6}}T_{\check{5}\check{4}}}T_{\check{2}\check{3}}T_{1\hat{2}}\\
&= \zeta^{12} T_{\check{5}6}T_{\check{5}1}T_{\check{5}\hat{3}}T_{\check{5}\hat{2}}T_{\check{2}3}\underline{T_{\check{2}4}}T_{\check{5}\hat{1}}T_{\check{5}\check{6}}\underline{T_{\check{5}\hat{2}}}T_{\check{2}\hat{6}}\underline{T_{\check{5}\check{4}}}T_{\check{5}\hat{2}}T_{\check{2}\hat{4}}T_{\check{2}\check{3}}T_{1\hat{2}}\\
&= \zeta^{12} \underline{T_{\check{5}6}T_{\check{5}1}T_{\check{5}\hat{3}}T_{\check{5}\hat{2}}}T_{\check{2}3}\underline{T_{\check{5}\hat{1}}T_{\check{5}\check{6}}T_{\check{5}\check{4}}}T_{\check{2}4}T_{\check{2}\hat{6}}T_{\check{5}\hat{2}}T_{\check{2}\hat{4}}T_{\check{2}\check{3}}T_{1\hat{2}} = \bar{\mathsf{F}}_f\bar{\mathsf{F}}_e,
\end{aligned}
$$

where each equality, except for the last one, is obtained by transforming the underlined fragment by applying the Pentagon relation (one time in the third, the fifth, the sixth, the seventh, the tenth, and three times in the ninth equalities), the Inversion relation (in the forth and the eighth equalities), and the extended symmetric group action (in the second, the forth, and the eighth equalities), while in the last equality the underlined (respectively the non-underlined) fragment corresponds to the operator $\bar{\mathsf{F}}_f$ (respectively $\bar{\mathsf{F}}_e$) $\square$

Lemma 2.7. *The lift of each puncture relation is ζ^6.*

Proof. Observe first that the central element P_i which is the lift of the puncture relation at the puncture p_i is independent on the particular subsurface $S_{0,3}^1$. If we consider another subsurface, there exists a homeomorphism $\varphi : S_{g,r}^s \to S_{g,r}^s$ fixing the puncture p_i and sending it to the initial subsurface, because the boundary components are non-separating. The new puncture relation is then conjugate of P_i by $\widetilde{\varphi}$ and hence they coincide, as they are elements of the center.

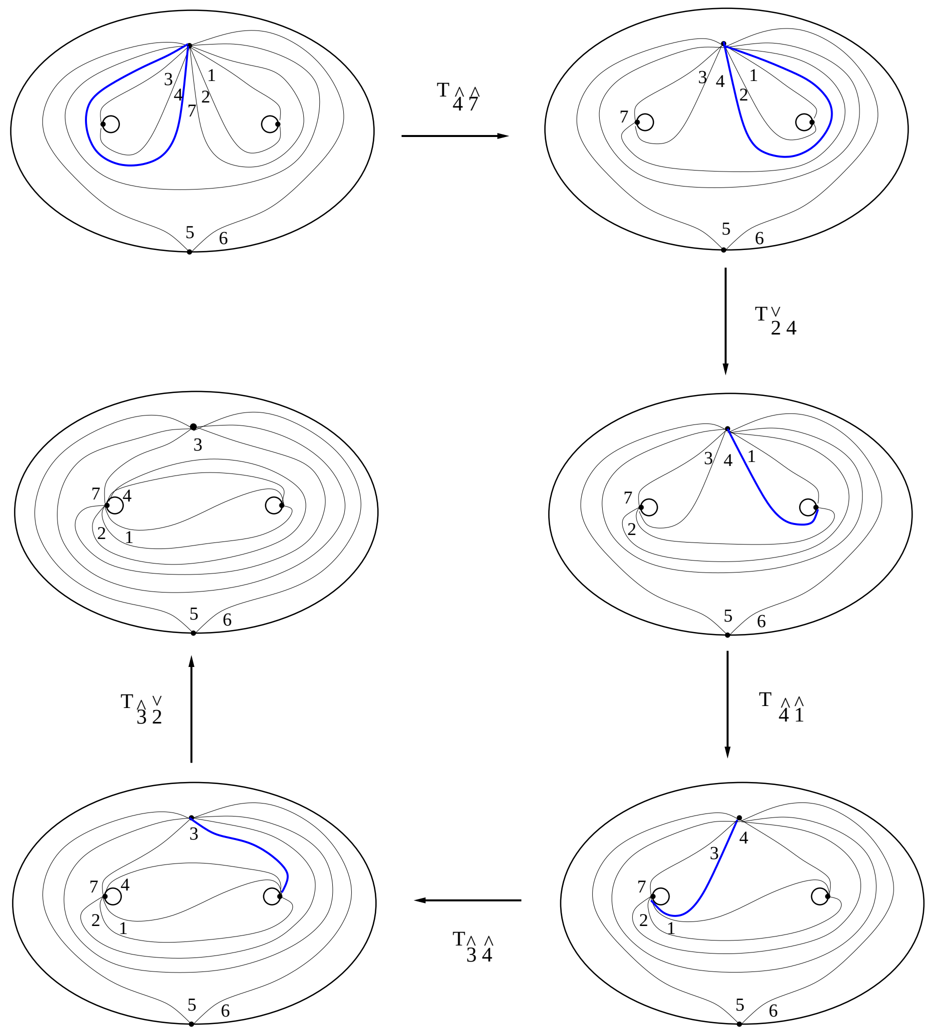
If $s \geq 4$ then there is an embedding $S_{0,3}^4 \subset S_{g,r}^s$, such that each boundary component of $S_{0,3}^4$ has a puncture on it. Consider first the following decomposition τ of the punctured pair of pants into triangles. The position of the label of each triangle indicates also the marked corner.



Then we can express easily the action of each Dehn twist D_{a_j} on the triangulation τ as a composition of flips. If we set $\mathsf{F}_{a_j} = \mathsf{F}(\tau, D_{a_j}(\tau))$ then we have:

$$\mathsf{F}_{a_1} = T_{\check{3}\check{4}}^{-1},\ \mathsf{F}_{a_2} = T_{\check{1}2}^{-1},\ \mathsf{F}_{a_3} = T_{\check{5}6}^{-1}$$

Further we use the sequence of transformations below, in order to change the triangulation τ into a triangulation which intersects the curve a_{12} in only two points.



Then the method outlined above permits to compute the Dehn twist $\mathsf{F}_{a_{12}} = \mathsf{F}(\tau, D_{a_{12}}(\tau))$ as follows:

$$\mathsf{F}_{a_{12}} = \mathrm{Ad}(T_{\hat4\hat7} T_{\check2 4} T_{\hat4\hat1} T_{\hat3\hat4} T_{\hat3\check2})(T^{-1}_{\check3\hat7})$$

Let us first simplify the formula for $\mathsf{F}_{a_{12}}$. We have

$$\begin{aligned}\bar{\mathsf{F}}_{a_{12}} &= T_{\check7 4} T_{\check2 4} T_{\hat4\hat1} T_{\hat3\hat4} \underline{T_{\hat3\check2} T_{\check7\hat3} \bar T_{\hat3\check2}} \bar T_{\hat3\hat4} \bar T_{\hat4\hat1} \bar T_{\check2 4} \bar T_{\check7 4} = T_{\check7 4} T_{\check2 4} T_{\hat4\hat1} \underline{T_{\hat3\hat4} T_{\check7\hat3}} T_{\check7\check2} \underline{\bar T_{\hat3\hat4}} \bar T_{\hat4\hat1} \bar T_{\check2 4} \bar T_{\check7 4} \\ &= T_{\check7 4} T_{\check2 4} \underline{T_{\hat4\hat1}} T_{\check7\hat3} \underline{T_{\check7\hat4}} T_{\check7\check2} \underline{\bar T_{\hat4\hat1}} \bar T_{\check2 4} \bar T_{\check7 4} = T_{\check7 4} T_{\check2 4} T_{\check7\hat3} T_{\check7\hat4} T_{\check7\hat1} \underline{T_{\check7\check2} \bar T_{\check2 4} \bar T_{\check7 4}} \\ &= T_{\check7 4} \underline{T_{\check2 4}} T_{\check7\hat3} \underline{T_{\check7\hat4}} T_{\check7\hat1} \underline{\bar T_{\check2 4}} T_{\check7\check2} = T_{\check7 4} T_{\check7\hat3} T_{\check7\hat4} T_{\check7\hat2} T_{\check7\hat1} T_{\check7\check2}\end{aligned}$$

where in each step the underlined fragment is transformed by using the Pentagon equation, and in the last equality it is also combined with the symmetry relation $T_{\check2 4} = T_{\hat4\hat2}$.

Our triangulation is invariant under the following simultaneous cyclic permutations

$$\pi\colon P_1 \mapsto P_2 \mapsto P_3 \mapsto P_1, \quad 1 \mapsto \check6 \mapsto 3 \mapsto 1, \quad 2 \mapsto \hat5 \mapsto \check4 \mapsto 2, \quad 7 \mapsto \check7,$$

so that the contours a_j and a_{kl} are transformed as follows:

$$\pi\colon a_1 \mapsto a_2 \mapsto a_3 \mapsto a_1, \quad a_{12} \mapsto a_{23} \mapsto a_{31} \mapsto a_{12}.$$

Thus, it suffices to know the explicit formula for $\mathsf{F}_{a_{12}}$ in order to write out the other two without any further calculation:

$$\bar{\mathsf{F}}_{a_{23}} = \pi(\bar{\mathsf{F}}_{a_{12}}) = \pi(T_{\check7 4} T_{\check7\hat3} T_{\check7\hat4} T_{\check7\hat2} T_{\check7\hat1} T_{\check7\check2}) = T_{\hat7\hat2} T_{\hat7\hat1} T_{\hat7\check2} T_{\hat7\check5} T_{\hat7 6} T_{\hat7 5},$$

and

$$\bar{\mathsf{F}}_{a_{31}} = \pi(\bar{\mathsf{F}}_{a_{23}}) = \pi(T_{\hat7\hat2} T_{\hat7\hat1} T_{\hat7\check2} T_{\hat7\check5} T_{\hat7 6} T_{\hat7 5}) = T_{7\check5} T_{76} T_{75} T_{74} T_{7\hat3} T_{7\hat4}.$$

Now, it remains to calculate the ξ in (1). We have

$$\begin{aligned}
\bar{\mathsf{F}}_{a_{12}}\bar{\mathsf{F}}_{a_{23}}\bar{\mathsf{F}}_{a_{31}} &= T_{\check{7}4}T_{\check{7}\hat{3}}T_{\check{7}\hat{4}}T_{\check{7}\hat{2}}T_{\check{7}\hat{1}}\underline{T_{\check{7}\check{2}}T_{\hat{7}\hat{2}}}T_{\hat{7}\hat{1}}T_{\hat{7}\check{2}}T_{\hat{7}\check{5}}T_{\hat{7}6}\underline{T_{\hat{7}5}T_{7\check{5}}}T_{76}T_{75}T_{74}T_{7\hat{3}}T_{7\hat{4}} \\
&= T_{\check{7}4}T_{\check{7}\hat{3}}T_{\check{7}\hat{4}}T_{\check{7}\hat{2}}T_{\check{7}\hat{1}}\zeta P_{(2\hat{7}\hat{2})}T_{\hat{7}\hat{1}}T_{\hat{7}\check{2}}T_{\hat{7}\check{5}}T_{\hat{7}6}\zeta P_{(\hat{5}7\check{5})}T_{76}T_{75}T_{74}T_{7\hat{3}}T_{7\hat{4}} \\
&= \zeta^2 T_{\check{7}4}T_{\check{7}\hat{3}}T_{\check{7}\hat{4}}\underline{T_{\check{7}\hat{2}}T_{\check{7}\hat{1}}T_{\hat{2}\hat{1}}}T_{\hat{2}7}\underline{T_{\hat{2}\check{5}}T_{\hat{2}6}T_{\check{5}6}}T_{\check{5}\check{2}}T_{\check{5}4}T_{\check{5}\hat{3}}T_{\check{5}\hat{4}}P_{(2\hat{7}5\check{2})} \\
&= \zeta^2 T_{\check{7}4}T_{\check{7}\hat{3}}T_{\check{7}\hat{4}}T_{\hat{2}\hat{1}}\underline{T_{\check{7}\hat{2}}T_{\hat{2}7}}T_{\check{5}6}\underline{T_{\hat{2}\check{5}}T_{\check{5}\check{2}}}T_{\check{5}4}T_{\check{5}\hat{3}}T_{\check{5}\hat{4}}P_{(2\hat{7}5\check{2})} \\
&= \zeta^2 T_{\check{7}4}T_{\check{7}\hat{3}}T_{\check{7}\hat{4}}T_{\hat{2}\hat{1}}\zeta P_{(\check{7}\hat{2}7)}T_{\check{5}6}\zeta P_{(\hat{2}\check{5}\check{2})}T_{\check{5}4}T_{\check{5}\hat{3}}T_{\check{5}\hat{4}}P_{(2\hat{7}5\check{2})} = \zeta^4 T_{\check{7}4}T_{\check{7}\hat{3}}\underline{T_{\check{7}\hat{4}}}T_{\hat{2}\hat{1}}T_{\check{5}6}\underline{T_{\hat{7}4}}T_{\hat{7}\hat{3}}T_{\hat{7}\hat{4}}P_{(7\hat{7})} \\
&= \zeta^4 T_{\check{7}4}T_{\check{7}\hat{3}}T_{\hat{2}\hat{1}}T_{\check{5}6}\zeta P_{(\check{4}\hat{7}4)}T_{\hat{7}\hat{3}}T_{\hat{7}\hat{4}}P_{(7\hat{7})} = \zeta^5 \underline{T_{\check{7}4}T_{\check{7}\hat{3}}}T_{\hat{2}\hat{1}}T_{\check{5}6}\underline{T_{4\hat{3}}}T_{47}P_{(74\check{7})} \\
&= \zeta^5 T_{\hat{2}\hat{1}}T_{\check{5}6}T_{4\hat{3}}\underline{T_{\check{7}4}T_{47}}P_{(74\check{7})} = \zeta^5 T_{\hat{2}\hat{1}}T_{\check{5}6}T_{4\hat{3}}\zeta P_{(\check{7}47)}P_{(74\check{7})} = \zeta^6 T_{\hat{2}\hat{1}}T_{\check{5}6}T_{4\hat{3}} = \zeta^6 \bar{\mathsf{F}}_{a_2}\bar{\mathsf{F}}_{a_3}\bar{\mathsf{F}}_{a_1}
\end{aligned}$$

where in the underlined fragments the Pentagon equation is used twice in the forth and once in the ninth equalities, the Inversion relation is used twice in the second and the fifth, and once in the seventh and the tenth equalities, while in the third, sixth, eighth, and eleventh equalities the permutation operators are moved to the right and the powers of the ζ, to the left.

$\square$

The following lemma is a simple consequence of a deep result of Gervais from ([10]):

Lemma 2.8. *Let $g \geq 2$ and $s \geq 0$. Then the group $\Gamma_{g,r}^s$ is presented as follows:*

1. *Generators are all Dehn twists D_a along the non-separating simple closed curves a on $\Sigma_{g,r}^s$.*
2. *Relations:*
   (a) *Braid-type 0 relations:*
   $$D_a D_b = D_b D_a$$
   for each pair of disjoint non-separating simple closed curves a and b;
   (b) *Braid type 1 relations:*
   $$D_a D_b D_a = D_b D_a D_b$$
   for each pair of non-separating simple closed curves a and b which intersect transversely in one point;
   (c) *One lantern relation for a 4-hold sphere embedded in $\Sigma_{g,r}^s$ so that all boundary curves are non-separating;*
   (d) *One chain relation for a 2-holed torus embedded in $\Sigma_{g,r}^s$ so that all boundary curves are non-separating;*
   (e) *A puncture relation for each puncture.*

Proof. According to ([10], Theorem B) we have a presentation of $\Gamma_{g,s+r}$ with the generators above and all but the puncture relations. Now, the kernel of $\Gamma_{g,s+r} \to \Gamma_{g,r}^s$ is the free Abelian group generated by the Dehn twists along the boundary curves to be pinched to punctures. Such a Dehn twist is expressed (using the lantern relation) by the left hand side of the puncture relation. This proves the claim. $\square$

Proof of Proposition 2.1. According to the normalization coming from the braid relations and the lantern relations the images of the standard Dehn twist generators of the mapping class group is a product of ζ^6 and elements T_{ij}, where i, j are the labels of the triangles (possibly with ˆ or ˇ). Thus the projective factors that appear belong to the subgroup A generated by ζ^6. The only non-trivial lift of a relation from Lemma 2.8 is the chain relation which lifts to ζ^{-72}. Set z for the element ζ^{-6} of $\widetilde{\Gamma_{g,r}^s}$. Then the presentation of the central extension $\widetilde{\Gamma_{g,r}^s}$ is given by the claimed relations.

2.3 Cohomological consequences

Recall from ([18], Corollary 4.4) that the 2-cohomology classes χ and e_i are defined for any $g \geq 3, s, r \geq 0$ and they span a free Abelian subgroup $\mathbb{Z}^{n+1} \subset H^2(\Gamma_{g,r}^s)$. This inclusion is actually an isomorphism when $g \geq 4$.

We will denote by $\widehat{\Gamma_{g,r}^s}$ the group defined by the presentation given in Proposition 2.1, for all values of s, g, r. Thus, according to Proposition 2.1 the extension $\widetilde{\Gamma_{g,r}^s}$ is isomorphic to $\widehat{\Gamma_{g,r}^s}$ if $s \geq 4$ and $g \geq 2$.

Lemma 2.9. *If $g \geq 2$, then we have $c_{\widehat{\Gamma_{g,r}}} = 12\chi \in H^2(\Gamma_{g,r}; A)$.*

Proof. Consider first the case where ζ is not a root of unity, so that the group A is isomorphic to $\mathbb{Z}$. Gervais proved in ([10], Theorem 3.6) that $\widehat{\Gamma_{g,r}}$ (namely, where $s = 0$) is isomorphic to the so-called p_1-central extension of $\Gamma_{g,r}$. Further in [10, 19] the authors identified the class of the p_1-central extension of $\Gamma_{g,r}$ to the class 12χ and thus $c_{\widehat{\Gamma_{g,r}}} = 12\chi$.

Here is a more direct argument. Set $\Gamma_{g,r}(1)$ for the subgroup of $\widehat{\Gamma_{g,r}}$ generated by the lifts $\widetilde{D_a}$ of the Dehn twists and the central element $u = z^{12}$. Then $\Gamma_{g,r}(1)$ is the universal central extension considered by Harer (see [10, 12]) and thus $c_{\Gamma_{g,r}(1)}$ is the generator χ of $H^2(\Gamma_{g,r}) \cong \mathbb{Z}$.

The cohomology class $c_{\Gamma_{g,r}(1)}$ is represented by some explicit 2-cocycle $C_{\Gamma_{g,r}(1)} : \Gamma_{g,r} \times \Gamma_{g,r} \to \mathbb{Z}$ which arises as follows. Let $S : \Gamma_{g,r} \to \Gamma_{g,r}(1)$ be a set-wise section. Let also $i : \ker(\Gamma_{g,r}(1) \to \Gamma_{g,r}) \to \mathbb{Z}$ be the group isomorphism defined by $i(u) = 1$. It is well-known that the 2-cocycle

$$C_{\Gamma_{g,r}(1)}(x, y) = i(S(xy)S(x)^{-1}S(y)^{-1}) \in \mathbb{Z}$$

represents the cohomology class $c_{\Gamma_{g,r}(1)}$.

Let us construct now a 2-cocycle representing the extension $\widehat{\Gamma_{g,r}}$. Consider the set-wise section $\iota \circ S : \Gamma_{g,r} \to \widehat{\Gamma_{g,r}}$, where $\iota : \Gamma_{g,r}(1) \to \widehat{\Gamma_{g,r}}$ is the obvious inclusion. Let also $j : \ker(\widehat{\Gamma_{g,r}} \to \Gamma_{g,r}) \to \mathbb{Z}$ be the isomorphism given by $j(z) = 1$. Then

$$C_{\widehat{\Gamma_{g,r}}}(x, y) = j((\iota \circ S)(xy)(\iota \circ S)(x)^{-1}(\iota \circ S)(y)^{-1}) = j(\iota(S(xy)S(x)^{-1}S(y)^{-1})) \in \mathbb{Z}$$

is a 2-cocycle representing $c_{\widehat{\Gamma_{g,r}}}$. Since $j(\iota(u)) = j(z^{12)} = 12i(u)$ and $S(xy)S(x)^{-1}S(y)^{-1}$ belongs to the cyclic subgroup of $\Gamma_{g,r}(1)$ generated by u, it follows that

$$C_{\widehat{\Gamma_{g,r}}}(x, y) = 12C_{\Gamma_{g,r}(1)}$$

and thus $c_{\widehat{\Gamma_{g,r}}} = 12\chi$, where χ is one fourth of the Meyer signature class, which is a generator of $H^2(\Gamma_{g,1}) \subset H^2(\Gamma_g^1)$.

When ζ is a root of unity of order N then the class of the extension $\widehat{\Gamma_{g,r}}$ is the image of 12χ in $H^2(\Gamma_{g,r}; \mathbb{Z}/N\mathbb{Z})$ by the reduction mod N. □

The next step is to prove a similar statement when the number s of punctures is non-zero.

Definition 2.4. *Let $\Gamma_{g,r}^s(a_1, \ldots, a_s)$ be the central extension of $\Gamma_{g,r}^s$ by A having the following presentation:*

1. *Generators are the $\widetilde{D_\alpha}$, where D_α are Dehn twist generators of $\Gamma_{g,r}^s$ and the central element z of the same order as ζ^{-6};*

2. *Relations are as follows. For each puncture p_i the lift of the corresponding puncture relation reads:*

$$\widetilde{D_{a_1(i)}}^{-1}\widetilde{D_{a_2(i)}}^{-1}\widetilde{D_{a_3(i)}}^{-1}\widetilde{D_{a_{12}(i)}}\widetilde{D_{a_{13}(i)}}\widetilde{D_{a_{23}(i)}} = z^{a_i}$$

where $\widetilde{D_a}$ are lifts of Dehn twists. Furthermore the chain and lantern relations have trivial lifts.

Proposition 2.2. *Suppose that $g \geq 0$. Then $c_{\Gamma_{g,r}^s(a_1,\ldots,a_s)} \in A^{n+1} \subset H^2(\Gamma_{g,r}^s; A)$ is the vector $a_1e_1 + a_2e_2 + \cdots + a_se_s$, where e_i is the Euler class of the i-th puncture.*

Proof. This is folklore. Consider first that ζ is not a root of unity. Let $\Sigma_{g,r+1;i}^{s-1}$ denote the subsurface of $\Sigma_{g,r}^s$ obtained by removing a one-punctured disk centered at the puncture p_i and thus creating a new boundary component b_i. We have then a central extension

$$\mathbb{Z} \to \Gamma_{g,r+1;i}^{s-1} \to \Gamma_{g,r}^s \to 1$$

induced by the inclusion map $\Sigma_{g,r+1;i}^{s-1} \hookrightarrow \Sigma_{g,r}^s$. It is well-known that its cohomology class is $c_{\Gamma_{g,r+1;i}^{s-1}} = e_i$.

Lemma 2.10. *The extension $\Gamma_{g,r+1;i}^{s-1}$ is isomorphic to $\Gamma_{g,r}^s(0, \ldots, 1, 0 \ldots, 0)$, where 1 is on the i-th position.*

Proof. There is a natural set-wise section $S_i : \Gamma^s_{g,r} \to \Gamma^{s-1}_{g,r+1;i}$, given by $S_i(D_\alpha) = D_\alpha$, for any Dehn twist D_α. In order to make sense, we might suppose that a simple closed curve α disjoint from the puncture p_i is actually disjoint from b_i so that it lies within $\Sigma^{s-1}_{g,r+1;i}$.

Braid, chain and lantern relations are then lifted trivially. A puncture relation at p_j is lifted trivially if $j \neq i$. Consider next a puncture relation at p_i in $\Sigma^s_{g,r}$, which is supported on some subsurface $\Sigma^1_{0,3}$. The three boundary curves of $\Sigma^1_{0,3}$ lie within $\Sigma^{s-1}_{g,r+1;i}$ and together with b_i bound a 4-holed sphere in $\Sigma^{s-1}_{g,r+1;i}$. The lantern relation associated to this 4-holed sphere on $\Sigma^{s-1}_{g,r+1;i}$ is then the lift of the puncture relation at p_i. The Dehn twist along b_i is the generator z of the central factor $\ker(\Gamma^{s-1}_{g,r+1;i} \to \Gamma^s_{g,r})$. Thus the lift of a puncture relation at p_i is the factor z. □

Lemma 2.11. *Let $L_{\mathbf{a}} : \mathbb{Z}^s \to \mathbb{Z}$ denote the linear map $L_{\mathbf{a}}(n_1, \ldots, n_s) = \sum_{i=1}^s a_i s_i$, where $\mathbf{a} = (a_1, \ldots, a_s)$. Consider the central extension*

$$1 \to \mathbb{Z}^s \to \Gamma_{g,r+s} \to \Gamma^s_{g,r} \to 1$$

Then the map $L_{\mathbf{a}}$ induces a quotient of $\Gamma_{g,r+s}$, which is a central extension $\Gamma^s_{g,r}(\mathbf{a})$ of $\Gamma^s_{g,r}$ by $\mathbb{Z}$ which is isomorphic to $\Gamma^s_{g,r}(a_1, \ldots, a_s)$ and gives rise to the following commutative diagram:

$$\begin{array}{ccccccccc}
1 & \to & \mathbb{Z}^s & \to & \Gamma_{g,r+s} & \to & \Gamma^s_{g,r} & \to & 1 \\
 & & \downarrow L_{\mathbf{a}} & & \downarrow \pi & & \downarrow \mathbf{1} & & \\
1 & \to & \mathbb{Z} & \to & \Gamma^s_{g,r}(\mathbf{a}) & \to & \Gamma^s_{g,r} & \to & 1
\end{array}$$

Proof. The class of the central extension $c_{\Gamma_{g,r+s}}$ belongs to $H^2(\Gamma^s_{g,r}; \mathbb{Z}^s) = \oplus_s H^2(\Gamma^s_{g,r}, \mathbb{Z})$. By functoriality we derive that $c_{\Gamma_{g,r+s}} = (e_1, e_2, \ldots, e_s) \in H^2(\Gamma^s_{g,r}; \mathbb{Z}^s)$. Then the class $c_{\Gamma^s_{g,r}(\mathbf{a})}$ is the image of $c_{\Gamma_{g,r+s}}$ into $H^2(\Gamma^s_{g,r}))$ by the homomorphism of coefficients rings $L_{\mathbf{a}} : \mathbb{Z}^s \to \mathbb{Z}$. There is an obvious set-wise section S defined in the same way as the S_i from above. Then $c_{\Gamma_{g,r+s}}$ is the class of the 2-cocycle $L_{\mathbf{a}}C$, where C is the 2-cocycle associated to S and so

$$L_{\mathbf{a}}C(x, y) = \pi(S(x)^{-1}S(y)^{-1}S(xy)) = L_{\mathbf{a}}((S_i(x)^{-1}S_i(y)^{-1}S_i(xy)_{i=1,s}) = \sum_{i=1}^{s} a_i C_i(x, y)$$

where C_i is the 2-cocycle associated to S_i. Since the class of C_i is e_i it follows that the class of $L_{\mathbf{a}}C$ is $\sum_{i=1}^s a_i ei$.

On the other hand the lifts of relations in $\Gamma^s_{g,r}(\mathbf{a})$ are the same as in $\Gamma^s_{g,r}(a_1, \ldots, a_s)$ and thus they are isomorphic. In fact the lifts of braid, chain and lantern relations to $\Gamma_{g,r+s}$ are trivial. The lift of a puncture relation at p_i is the i-th generator of the central factor $\mathbb{Z}^s$, according to Lemma 2.10. Therefore its image into $\Gamma^s_{g,r}(\mathbf{a})$ is z_i^a, namely the lift of the puncture relation in $\Gamma^s_{g,r}(a_1, \ldots, a_s)$. □

When ζ is a root of unity the extensions by $\mathbb{Z}$ above are replaced by extensions by $\mathbb{Z}/N\mathbb{Z}$ and all arguments go through without essential modifications.

This proves the Proposition. □

Proof of the Theorem. Assume first that A is cyclic infinite. Let f denote the surjective homomorphism $f : \Gamma^s_{g,r} \to \Gamma_{g,r}$. Consider the central extension

$$1 \to \mathbb{Z}^2 \to f^*(\widehat{\Gamma_{g,r}}) \times \Gamma^s_{g,r}(1, 1, \ldots, 1) \to \Gamma^s_{g,r} \to 1$$

Using the map $L : \mathbb{Z}^2 \to \mathbb{Z}$ given by $L(x, y) = x + y$ we find a quotient of $f^*(\widehat{\Gamma_{g,r}}) \times \Gamma^s_{g,r}(1, 1, \ldots, 1)$, which is a central extension by $\mathbb{Z}$ isomorphic to $\widehat{\Gamma^s_{g,r}}$. In fact, there is a commutative diagram:

$$\begin{array}{ccccccccc}
1 & \to & \mathbb{Z}^2 & \to & f^*(\widehat{\Gamma_{g,r}}) \times \Gamma^s_{g,r}(1, 1, \ldots, 1) & \to & \Gamma^s_{g,r} & \to & 1 \\
 & & \downarrow L & & \downarrow \pi & & \downarrow \mathbf{1} & & \\
1 & \to & \mathbb{Z} & \to & \widehat{\Gamma^s_{g,r}} & \to & \Gamma^s_{g,r} & \to & 1
\end{array}$$

The central extension from the lower row is isomorphic to $\widehat{\Gamma^s_{g,r}}$ because the lifts of relations are the same. Braid and lantern relations lift trivially. Chain relations lift to z^{12} in $f^*(\widehat{\Gamma_{g,r}})$ and trivially to $\Gamma^s_{g,r}(1, 1, \ldots, 1)$ and thus the image of the lift by L (or π) is z^{12}. Puncture relations at p_i lift trivially to $f^*(\widehat{\Gamma_{g,r}})$ and to z in the factor $\Gamma^s_{g,r}(1, 1, \ldots, 1)$, so that its image by L (or π) is z. As a consequence of this description the class $c_{\widehat{\Gamma^s_{g,r}}}$ is the image by L of the class of $f^*(\widehat{\Gamma_{g,r}}) \times \Gamma^s_{g,r}(1, 1, \ldots, 1)$, namely $c_{f^*(\widehat{\Gamma_{g,r}}} + c_{\Gamma^s_{g,r}(1,1,\ldots,1)}$

On the other hand, by functoriality, the class $c_{f^*(\widehat{\Gamma_{g,r}})}$ is $f^*(12\chi) = 12\chi \in H^2(\Gamma_{g,r}^s)$, because the map f^* is the standard embedding of $H^2(\Gamma_{g,r}) = \mathbb{Z}\chi$ into $H^2(\Gamma_{g,r}^s)$. Proposition 2.2 proves the Theorem for $g \geq 3$.

When $g = 2$ one does not know the $H^2(\Gamma_{2,r}^s)$, but for $s = 0$ and $r \leq 1$. Nevertheless the classes χ and e_j are still defined. It suffices to prove that:

Lemma 2.12. *The subgroup of $H^2(\Gamma_{2,r}^s)$ generated by χ and $e_1, \ldots, e_s$ is isomorphic to $\mathbb{Z}/10\mathbb{Z} \oplus \mathbb{Z}^s$.*

Proof. By the universal coefficients theorem we have

$$1 \to H_1(\Gamma_{2,r}^s) \to H^2(\Gamma_{2,r}^s) \to \mathrm{Hom}(H_2(\Gamma_{2,r}^s), \mathbb{Z}) \to 1$$

From ([18], Proposition 1.6) we have $H_1(\Gamma_{2,r}^s) = \mathbb{Z}/10\mathbb{Z}$. The Meyer class χ in genus 2 is one half of the class of Meyer's cocycle from [20] and it generates the image of $H_1(\Gamma_{2,r}^s)$ into $H^2(\Gamma_{2,r}^s)$.

Consider next the extensions $\Gamma_{2,r}^s(\mathbf{a})$ for integral vectors $\mathbf{a}$. According to the previous description lifts of puncture relations are of the form z^{a_i}. Suppose that there exists an isomorphism between the extensions $\Gamma_{2,r}^s(\mathbf{a})$ and $\Gamma_{2,r}^s(\mathbf{b})$. Such an isomorphism of extensions should send $\widetilde{D_\alpha}$ into $z^{n(\alpha)}\widetilde{D_\alpha}$, because it has to induce identity on $\Gamma_{2,r}^s$. Since lifts of braid relations are trivial in both extension groups it follows that $n(\alpha) = n$ does not depend on the non-separating curve α. But puncture relations are homogeneous, and so they are therefore independent do not depend on n. This shows that $\mathbf{a} = \mathbf{b}$. In particular the classes e_i span a free $\mathbb{Z}$-submodule of $H^2(\Gamma_{2,r}^s)$.

Since the class χ is of order 10 and both subgroups $\mathbb{Z}/10\mathbb{Z}$ (generated by χ) and $\mathbb{Z}^s$ (generated by $e_1, \ldots, e_s$) inject into $H^2(\Gamma_{2,r}^s)$, the claim follows. $\square$

Then the arguments used above for $g \geq 3$ work as well for $g = 2$ and the Theorem follows. When ζ is a root of unity the associated cohomology class is the reduction mod N of the corresponding integral cohomology class.

Proof of Corollary 0.2. Consider the extension $\widehat{\Gamma_{g,r+s}}$ of class 12χ. The Corollary claims that there is an exact sequence:

$$1 \to A^{s-1} \to \widehat{\Gamma_{g,r+s}} \to \widehat{\Gamma_{g,r}^s} \to 1$$

This can be verified by using the explicit presentations of the two groups involved. The kernel is generated by the products of two opposite Dehn twists on the s blown up boundary components.

Proof of Corollary 0.3. It suffices to understand the map $H^2(\Gamma_{g,r}^s; A) \to H^2(\Gamma_{g,r}^s, \mathbb{C}^*)$ induced by $z \to \zeta^{-6}$. This map is injective, when $g \geq 3$.

The Universal Coefficients Theorem states that, for any Abelian group W, the following exact sequence is exact:

$$1 \to \mathrm{Ext}(H_0(\Gamma_{g,r}^s), W) \to H^1(\Gamma_{g,r}^s; W) \to \mathrm{Hom}(H_1(\Gamma_{g,r}^s), W) \to 1$$

Now $\mathrm{Ext}(\mathbb{Z}, W) = 0$, for any Abelian group W. This implies that $H^1(\Gamma_{g,r}^s; \mathbb{C}^*) = H^1(\Gamma_{g,r}^s; \mathbb{C}^*/A) = 0$, if $g \geq 3$. From the Bockstein exact sequence

$$H^1(\Gamma_{g,r}^s; \mathbb{C}^*) \to H^1(\Gamma_{g,r}^s; \mathbb{C}^*/A) \xrightarrow{\beta} H^2(\Gamma_{g,r}^s; A) \xrightarrow{\nu} H^2(\Gamma_{g,r}^s; \mathbb{C}^*)$$

we derive the claim.

When $g = 2$ the Universal Coefficient Theorem shows, as above, that $H^1(\Gamma_{2,r}^s; \mathbb{C}^*) = \mathrm{Hom}(H_1(\Gamma_{2,r}^s), \mathbb{C}^*)$ and $H^1(\Gamma_{2,r}^s; \mathbb{C}^*/A) = \mathrm{Hom}(H_1(\Gamma_{2,r}^s), \mathbb{C}^*/A)$. Thus $H^1(\Gamma_{2,r}^s; \mathbb{C}^*) = \mathrm{Hom}(\mathbb{Z}/10\mathbb{Z}, \mathbb{C}^*) = U_{10}$, where U_{10} is the subgroup of roots of unity of order 10. The last isomorphism sends a homomorphism into its value on the generator 1. Next $H^1(\Gamma_{2,r}^s; \mathbb{C}^*/A) = \mathrm{Hom}(\mathbb{Z}/10\mathbb{Z}, C^*/A) = U_{10} \times A/10A$. To explain the last isomorphism, each element $f \in H^1(\Gamma_{2,r}^s; \mathbb{C}^*/A)$ is determined by its value $f(1) = As$, for some $s \in \mathbb{C}^*$. Here $s^{10} = a^n \in A$, where a is the generator of A. Fix some 10-th root $a^{1/10} \in \mathbb{C}^*$ of the generator of A. Then the isomorphism above associates to f the element $(sa^{-n/10}, s^{10}) \in U_{10} \times A/10A$, which is well-defined and independent of the choice of the representative s in its A-coset. In particular the map $H^1(\Gamma_{2,r}^s, \mathbb{C}^*) \to H^1(\Gamma_{2,r}^s, \mathbb{C}^*/A)$ sends U_{10} onto the factor U_{10} of the second group.

Let $\widehat{f}$ be a lift of f to $\widehat{f} : \mathbb{Z}/10\mathbb{Z} = H_1(\Gamma_{g,r}^s) \to \mathbb{C}^*$, for instance $\widehat{f}(k) = s^k$, where $k \in \mathbb{Z}/10\mathbb{Z}$. Then $F(k_1, k_2) = \widehat{f}(k_1)\widehat{f}(k_2)\widehat{f}(k_1k_2)^{-1} \in A$ is a 2-cocycle on $H_1(\Gamma_{2,r}^s)$ with values in A. The pull-back in $H^2(\Gamma_{2,r}^s, A)$ of the class of F by the map $\Gamma_{2,r}^s \to H_1(\Gamma_{2,r}^s)$ is the element $\beta(f)$. It is well-known that $H^2(\mathbb{Z}/10\mathbb{Z}, A) = A/10A$ is generated by the Euler class. Specifically, the cohomology class of the 2-cocycle F in $H^2(\mathbb{Z}/10\mathbb{Z}, A)$ is the element $s^{10} \in A/10A$, under the previous isomorphism.

The Universal Coefficients Theorem shows that

$$1 \to \mathrm{Ext}(H_1(\Gamma^s_{2,r}), A) \to H^2(\Gamma^s_{2,r}; A) \to \mathrm{Hom}(H_2(\Gamma^s_{2,r}), A) \to 1$$

Further $\mathrm{Ext}(H_1(\Gamma^s_{g,r}), A) = A/10A$ is generated by the class χ (as an A-valued cohomology class). Using the definition of Ext one identifies the class χ with the generator of $H^2(\mathbb{Z}/10\mathbb{Z}; A)$. This implies that the image of β is the subgroup generated by χ within $H^2(\Gamma^s_{2,r}), A)$. Then Corollary 0.3 follows.

References

[1] S. Baseilhac and R. Benedetti, *Classical and quantum dilogarithmic invariants of flat PSL(2,C)-bundles over 3-manifolds*, Geometry & Topology 9 (2005), 493-569.

[2] Hua Bai, F. Bonahon and Xiaobo Liu, *Local representations of the quantum Teichmüller space*, math.GT/0707.2151.

[3] F. Bonahon and Xiaobo Liu, *Representations of the quantum Teichmüller space and invariants of surfaces diffeomorphisms*, Geometry & Topology 11 (2007), 889–937.

[4] L. Chekhov and V. Fock, *Quantum Teichmüller space*, math.QA/9908165.

[5] L.D. Faddeev, *Discrete Heisenberg-Weyl group and modular group*, Lett. Math. Phys. 34 (1995), 249-254.

[6] V. Fock, *Dual Teichmúller spaces*, math.dg-ga/9702018.

[7] V. Fock and A.B. Goncharov, *Moduli spaces of local systems and higher Teichmüller theory*, Inst. Hautes Études Sci. Publ. Math. no.103 (2006), 1-211.

[8] V. Fock and A.B. Goncharov, *The quantum dilogarithm and representations of quantum cluster varieties*, Inventiones Math. 175 (2009), 223–286.

[9] L. Funar and V. Sergiescu, *Central extensions of the Ptolemy-Thompson group and the quantized Teichmüller theory*, J. Topology, to appear, arXiv:0802.2996.

[10] S. Gervais, *Presentation and central extensions of mapping class groups*, Trans. Amer. Math. Soc. 348 (1996), no. 8, 3097–3132.

[11] S. Gervais, *A finite presentation of the mapping class group of a punctured surface*, Topology 40(2001), 703–725.

[12] J. Harer, *The second homology group of the mapping class group of an orientable surface*, Invent. Math. 72 (1983), no. 2, 221–239.

[13] R.M. Kashaev, *Quantum Dilogarithm as a 6j-Symbol*, Mod. Phys. Lett. A9 (1994), 3757–3768, arXiv:hep-th/9411147.

[14] R.M. Kashaev, *Quantum hyperbolic invariants of knots*, Oxford lecture series in mathematics and its applications, vol. 16 (1999), 343–360.

[15] R.M. Kashaev, *Quantization of Teichmüller spaces and the quantum dilogarithm*, Lett. Math. Phys. 43 (1998), 105-115.

[16] R.M. Kashaev, *The Liouville central charge in quantum Teichmüller theory*, Tr. Mat. Inst. Steklova 226 (1999), Mat. Fiz. Probl. Kvantovoi Teor. Polya, 72–81; translation in Proc. Steklov Inst. Math., no. 3 (226) (1999), 63–71.

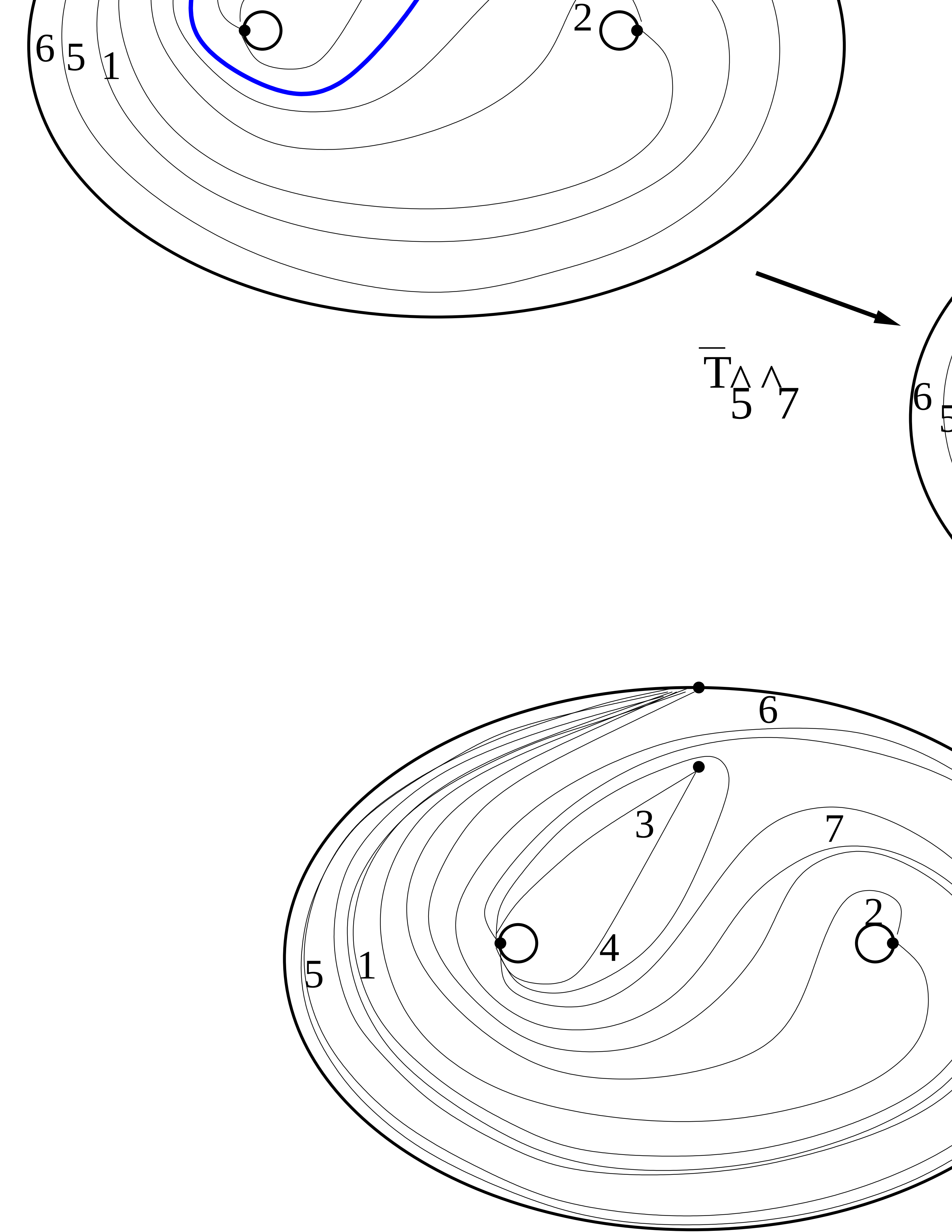
[17] M. Korkmaz, Low-dimensional homology groups of mapping class groups: a survey. Turkish J. Math. 26 no. 1 (2002), 101–114.

[18] M. Korkmaz and A.I. Stipsicz, *The second homology groups of mapping class groups of oriented surfaces*, Math. Proc. Cambridge Philos. Soc. 134 (2003), 479–489.

[19] G. Masbaum, J. Roberts, *On central extensions of mapping class groups*, Math. Ann. 302, no. 1 (1995), 131–150.

[20] W.Meyer, *Die Signatur von Flchenbndeln*, Math. Ann. 201 (1973), 239–264.

[21] L. Paris and D. Rolfsen, *Geometric subgroups of mapping class groups*, J. Reine Angew. Math. 521 (2000), 47–83.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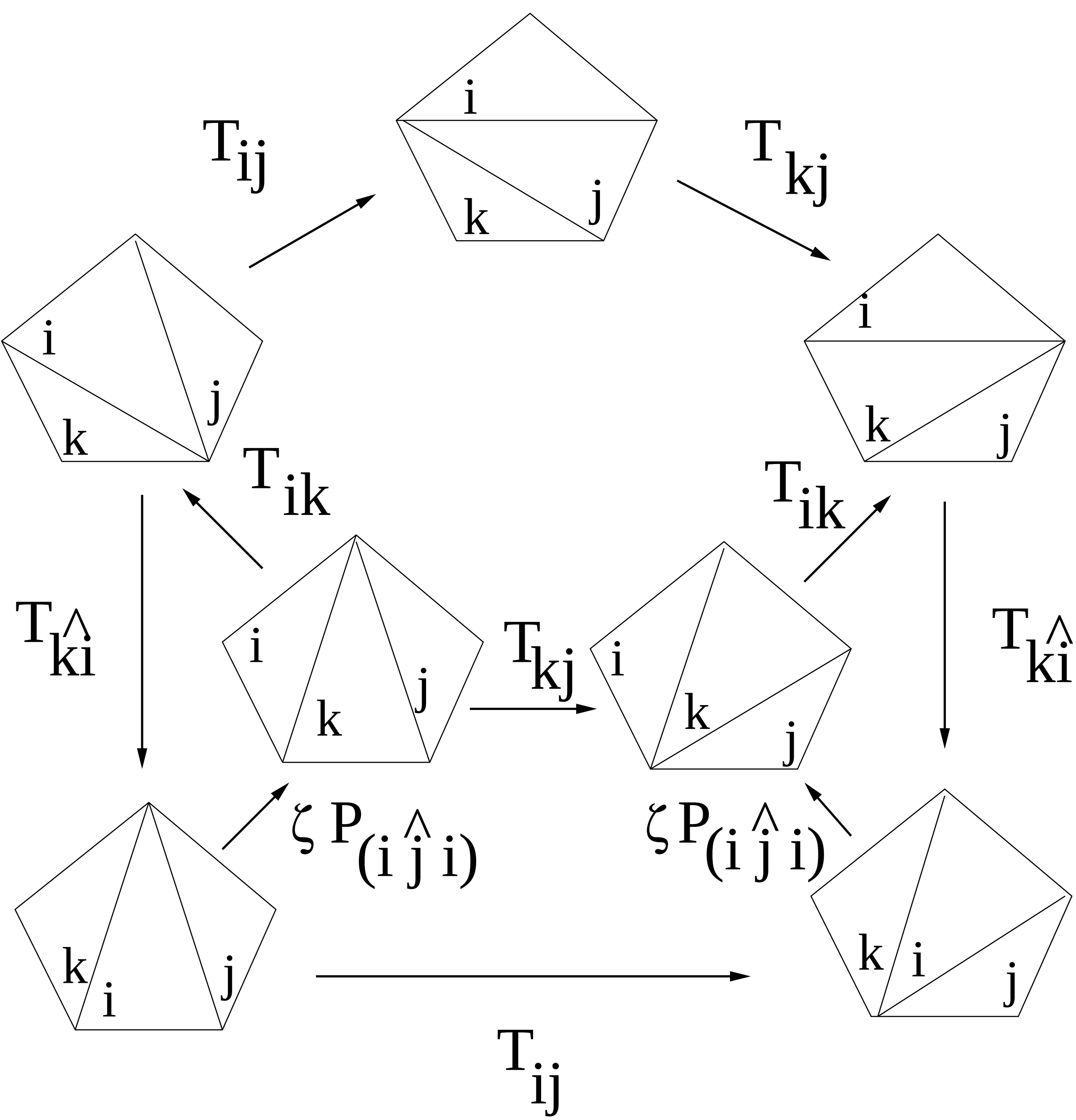

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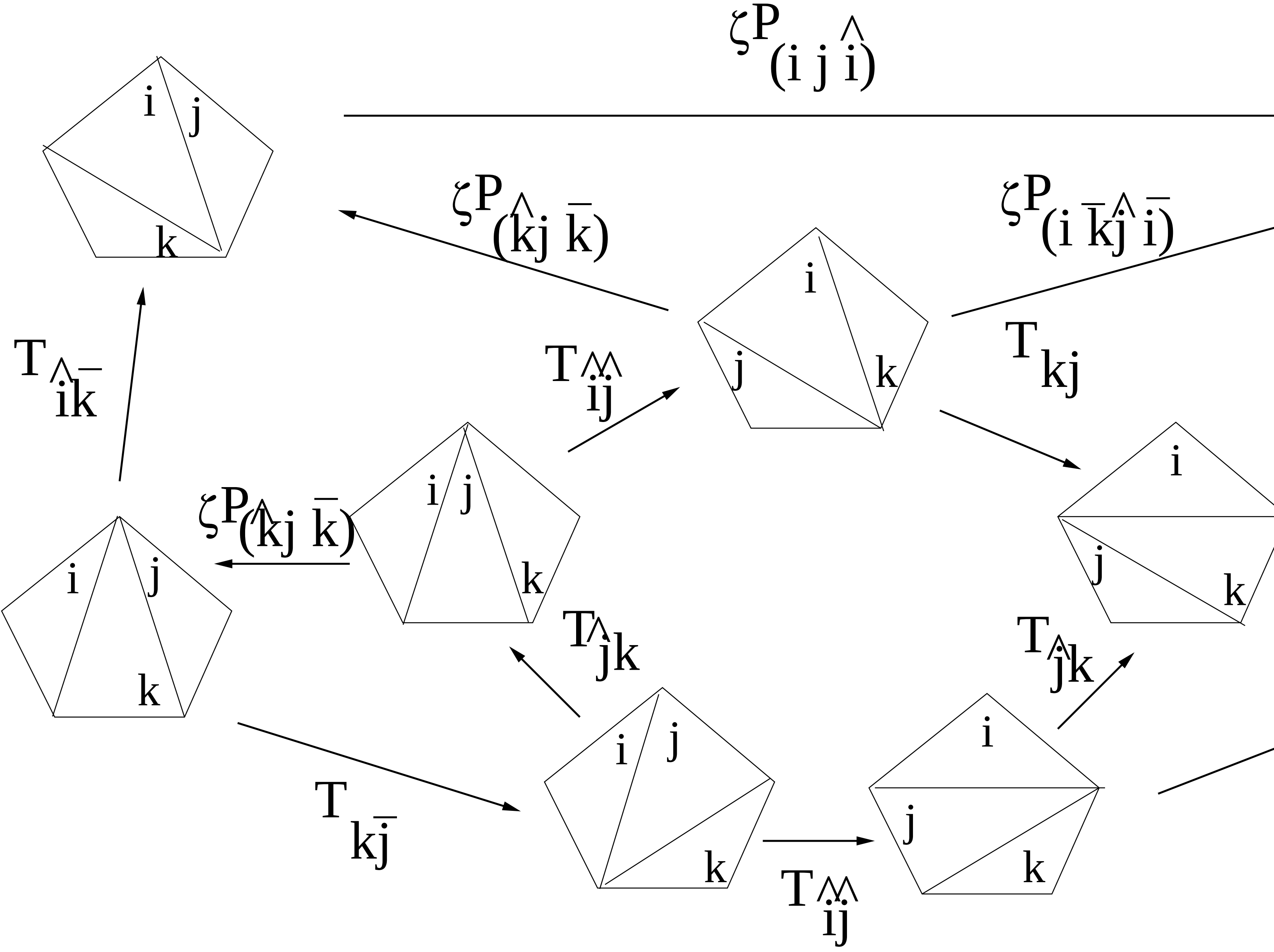

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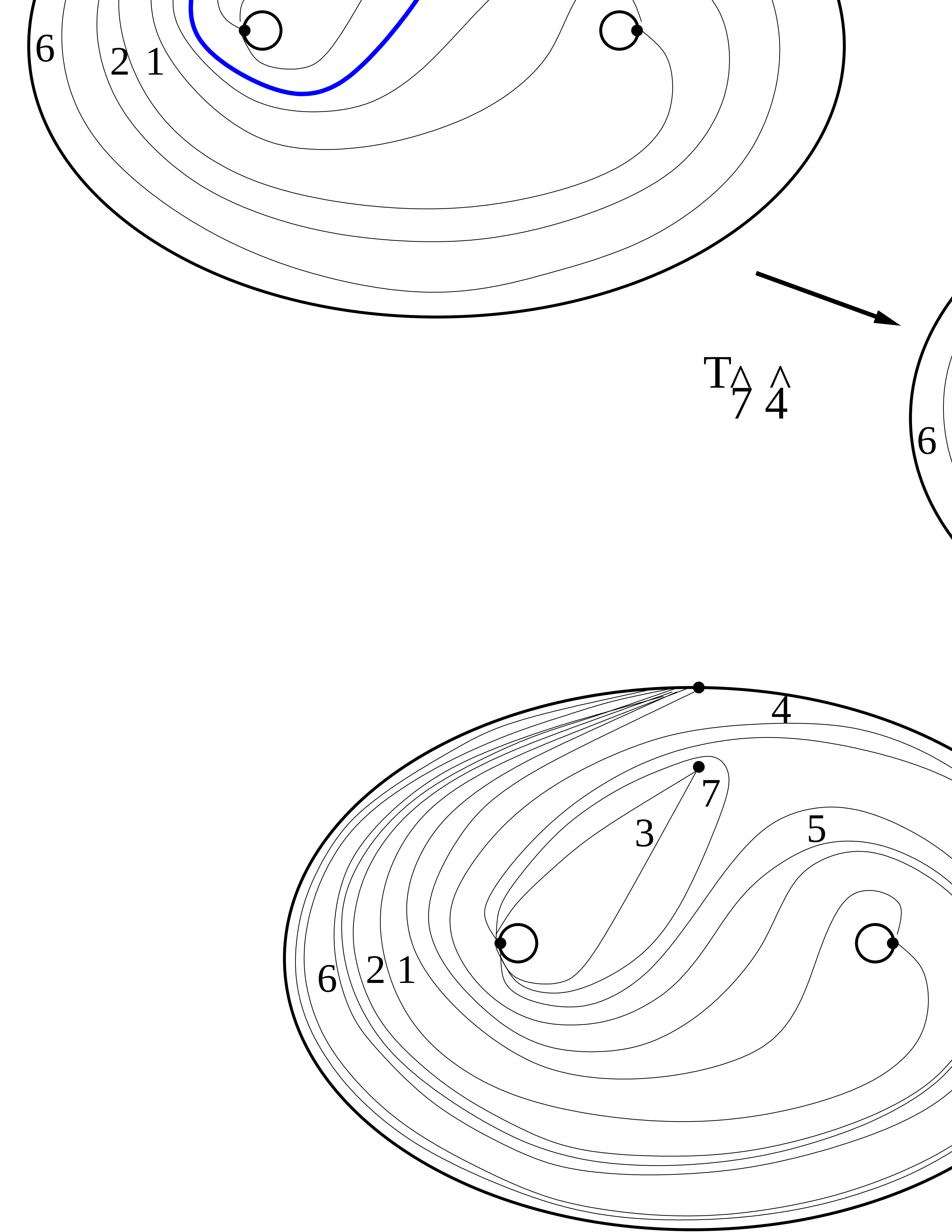

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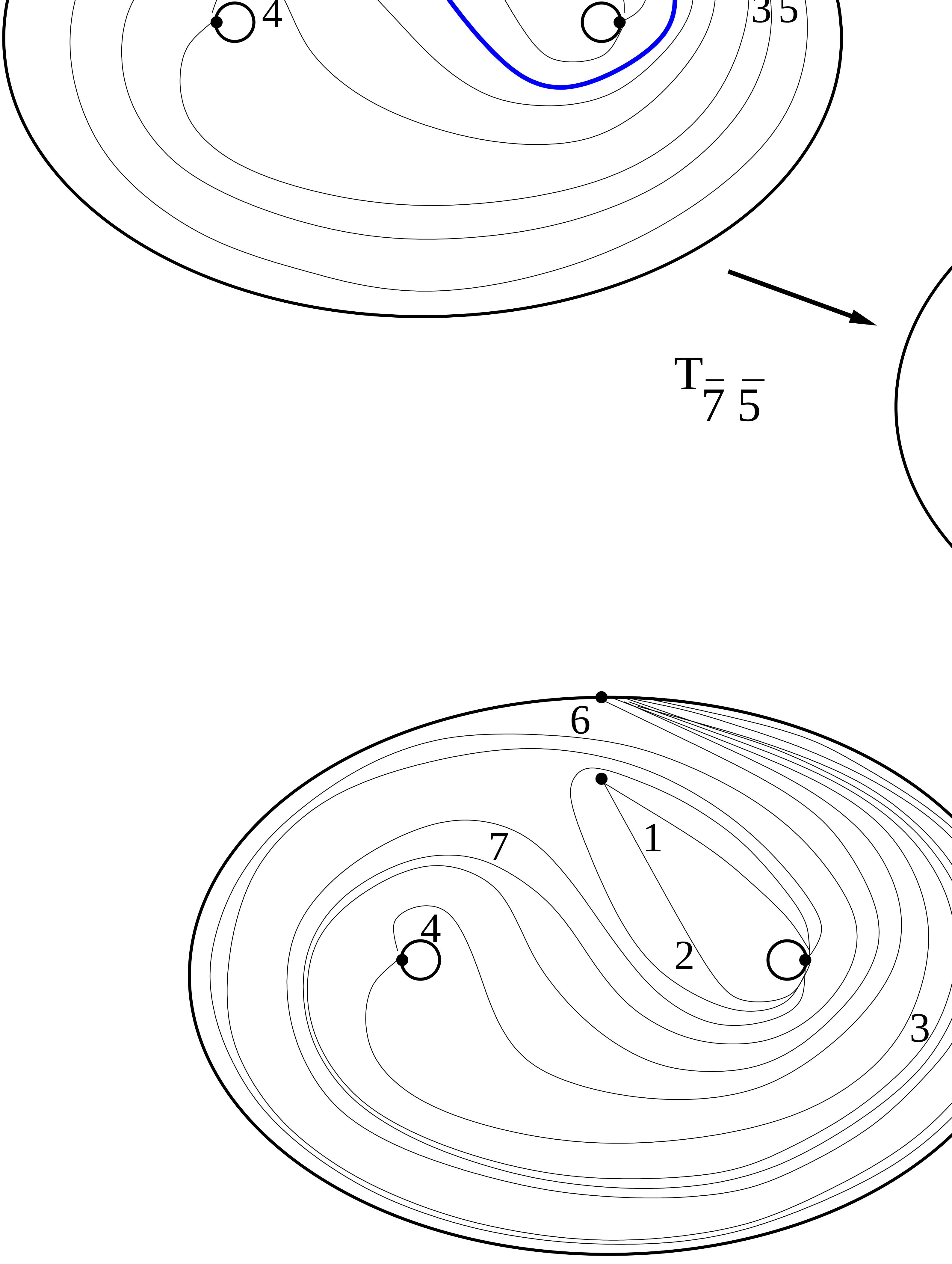

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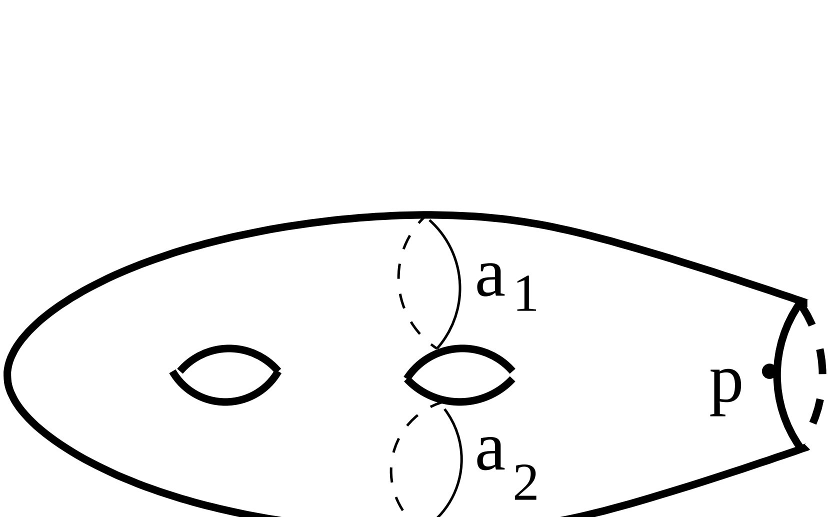
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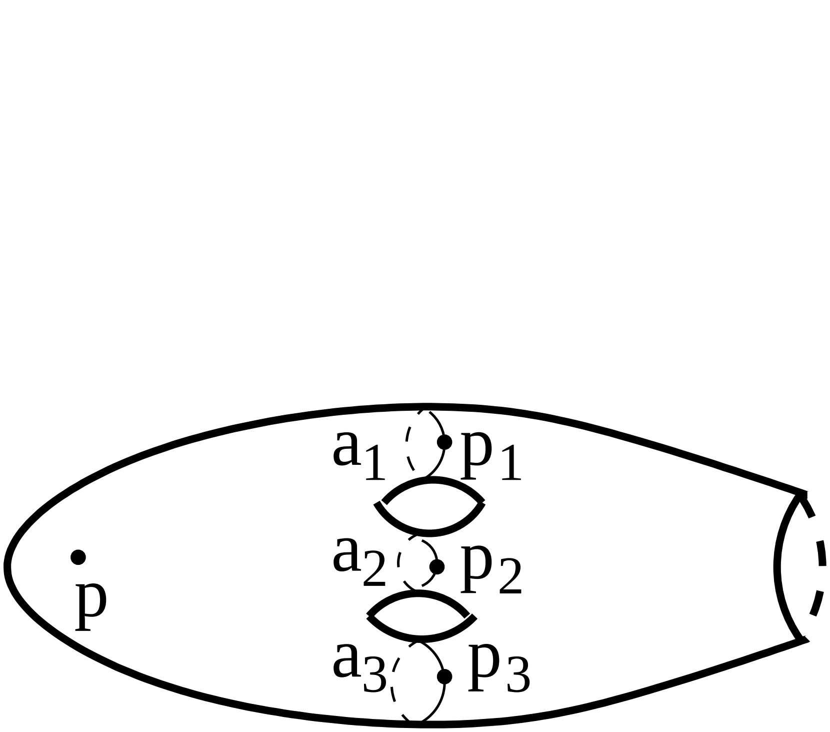
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